The Theory of Vying for Dominance in Dynamic Network Formation

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Abstract

In many networks, a few highly central nodes have out-sized impacts on the behavior of the system and generate a large amount of value from their position, but what determines which nodes become central? We hypothesize that the timing of entry into the network can play a critical role. In this paper, we present a new dynamic model of network formation with history dependence, growth, and forward looking strategic agents. These features can generate novel strategic behaviors such as "vying for dominance," whereby an individual makes many connections as he joins the network because he expects doing so will attract more connections from later nodes. We find that all players either vie for dominance or play myopically in equilibrium. Furthermore, if we assume players use a novelty seeking tie-breaking rule, the solution is characterized by periodic vying for dominance separated by periods of low connection, myopic play. Because vying becomes more expensive as the network grows, the time between profitable vying opportunities increases exponentially over time, and the network becomes more stagnant.

1 Introduction

The structure of the networks underlying economic interactions can have substantial impacts on the outcomes of economic systems. The theoretical literature provides many examples where network structure plays a large role in determining the behavior of agents in economically important settings such as trading networks,[1] coordination games,[2] public goods provision,[3] and markets with network externalities.[4] Empirically, the structure of real-world networks has been found to have significant impacts on many interesting features of economic systems, such as market volatility, job market efficiency, and technological diffusion.[5] Among commonly studied network features, centrality is considered to be a particularly valuable and important component of network structure, providing information, bargaining power, and influence.[6]

The considerable importance of centrality leads us to consider how some nodes become central when networks form. In this paper we explore the novel hypothesis that the order in which nodes join a network can play a critical role in determining how some nodes become central. Existing dynamic models of network formation cannot be used to answer this question because they do not allow for dynamic entry.[7] Because the set of possible outcome networks can be defined based on static features of the network, these existing models are generally not well suited to answer questions about how dynamic features of the network formation process, like entry order and technological change, affect

[1] Corominas-Bosch (2004); Kranton and Minehart (2001); and Blume et al. (2009)

[2] Calvó-Armengol et al. (2015); Apt et al. (2016); and McCubbins and Weller (2012)

[3] Allouch (2015); Bramoullé and Kranton (2007); Bourlès et al. (2017); and Carpenter et al. (2012)

[4] Candogan et al. (2012)

[5] For several reviews of the literature on the importance of network structure in empirical settings see: Bala and Goyal (2000); Jackson (2003); Jackson and Wolinsky (1996); andCarrillo and Gaduh (2012)

[6] Theoretical studies include Kranton and Minehart (2001); Blume et al. (2009); Apt et al. (2016); Chen and Teng (2016). Empirical studies include Pollack et al. (2015); Sarigöl et al. (2014); Powell et al. (1996); Rossi et al. (2015)

[7] Bala and Goyal (2000); Watts (2001); Currarini and Morelli (2000); Mutuswami and Winter (2002); and Song and van der Schaar (2015)

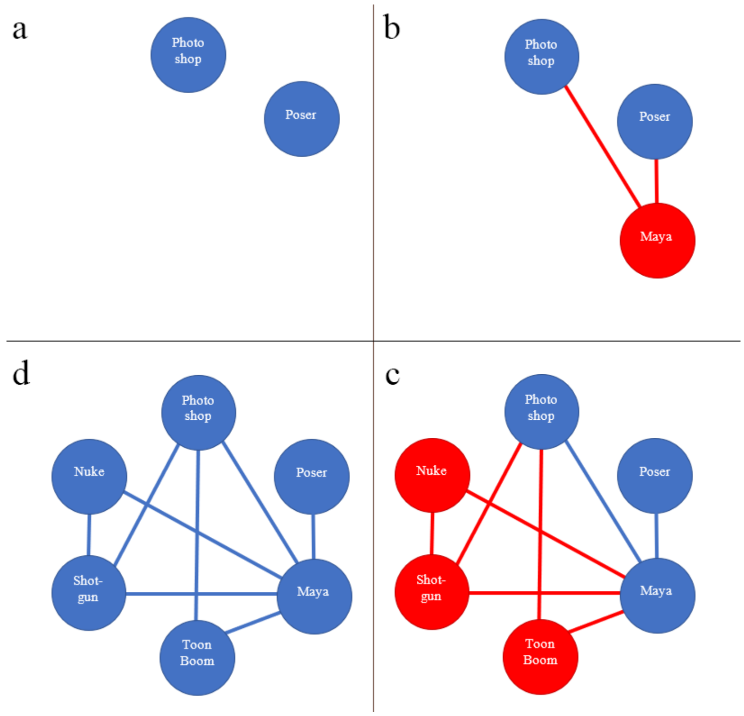

Figure 1: The formation of a network of compatibilities in animation software starting at the top left and going clockwise. Red denotes new nodes and connections.

network structure. For example, in the model of Currarini and Morelli (2000) all efficient networks can form, and in the Watts (2001) model all pairwise stable networks are solutions.

Therefore, we need a new, dynamically strategic model of network growth which can account for the relationship between entry timing and centrality. Our novel model will help explain why many networks of interest are dominated by early movers but not first movers. For example, consider the video game market, where relatively early movers like Nintendo dominate but even earlier movers like Atari are largely absent at this time. In our model, nodes joining the network at specific key times will have the opportunity to attempt to become central by choosing to vie for dominance, a type of move where the player makes many connections as he joins the network in order to become one of the most central nodes in the network. Being highly central can provide benefits in the form of free connections from nodes joining the network later on. Generally these vying opportunities will arise early in the network formation process but not at the beginning.

To provide some intuition about how entry timing influences node centrality, consider an example borrowed from the accompanying experimental paper, Neligh (2019). This example also shows how economic systems can come to be dominated by early movers but not necessarily by first movers. Animation software is a useful example because most animation projects require several pieces of software to complete. This makes compatibility and centrality important to the developers. In addition, whether two pieces of software are compatible is generally determined by the developer of the newer software, which simplifies the logic. Figure 1 illustrates the formation of a network composed of animation softwares, where edges represent compatibilities in the form of shared file types or plugins that let the user easily move work from one piece of software to another.

The network begins with Photoshop and Poser unconnected (Figure 1a). Maya enters the network and establishes connections (representing compatibilities) with the two existing nodes (Figure 1b). Additional nodes representing Toon Boom, Shotgun, and Nuke join the network and connect to Maya (Figure 1c). In the final network, Maya has a very central position with connections to all five other nodes (Figure 1d). A node is considered *dominant* if it is one of the most central nodes in the network.

Why do players form the network in this way? We propose that their behavior arises naturally from the fact that centrality is beneficial but connections are costly. Hence, later players would want to connect to the set of nodes which provides them as much centrality as possible with a minimum number of connections. Generally this involves making a connection to the most central (i.e. dominant) node in the network,[8] because making such a connection allows the player to inherit some of the centrality of that node. The last three nodes can be seen as choosing this type of myopic move. We call a move *myopic* if it would be the optimal choice for a version of the game that ended immediately afterwards. In the games we discuss in this paper, this means making one connection to one of the most central (or *dominant*) nodes.

Note that myopic moves in this example lead to a form of preferential attachment, which is a property whereby having a relatively high centrality causes a node's centrality to increase more quickly over time. This phenomenon was first observed in taxonomic networks by Yule (1925), but it has since been found to be a common feature of networks, arising in transaction networks,[9] social networks,[10] online link networks,[11] scientific collaboration networks,[12] and citation networks.[13]

However, if preferential attachment were the only force at play, we would expect the most-connected nodes to be the first joiners, rather than nodes like Maya that entered later. Furthermore, Maya's move was qualitatively different from the moves of later nodes. Maya made connections to all existing nodes and become the most central node in the network, i.e. Maya vied for dominance. We say a player *vies for dominance* when his move causes him to immediately become dominant.

Because Maya became the most central node in the network, it gained connections from the later three myopic players. This future benefit means that vying for dominance can be optimal even if it requires a large investment in current connections.

The question still remains of why Maya would vie for dominance while other nodes did not. For nodes that joined after Maya, vying for dominance would have been too expensive, requiring more connections and hence more time and money. Furthermore, the rewards would have been lower because as time goes on, the number of possible future compatibilities decreases. For the players who joined the network before Maya, the maximum achievable centrality would be restricted, limiting the value of dominance. Also, their rational anticipation of Maya's vying would make their own vying less valuable.

Maya was in a sweet spot: there were enough nodes present to allow Maya to effectively vie for dominance but not so many that vying was prohibitively expensive.

This example suggests that a network formation model should include a number of dynamic features: history dependence, growth, and forward looking strategic agents. These features are also realistic in many systems of interest. Most networks of interest display history dependence with the continuing evolution of the network depending non-trivially on its current state.[14] In addition, many networks expand with new nodes accumulating over time.[15] The existence of forward looking, strategic agents enables the kind of strategically rich behavior exhibited by Maya in the example.

This paper has two main messages which carry through all sections. The first message is that vying for dominance is an important phenomenon to consider when dealing with networks where connections are stable, centrality is beneficial, and newer nodes tend to sponsor their connections to older nodes. High centrality now can lead to cheap or free connections in the future, which means it can be profitable to invest heavily in centrality early on.

The second message is that networks tend to be dominated by nodes that arrived early in the network formation process, but not at the very beginning, due to the way that profitable vying opportunities are distributed in time. The cost of vying tends to grow as the network grows, and the benefits of strategic action shrink approaching the end of the game, while the costs and benefits of myopic behavior remain stable. As a result networks end up being dominated by early movers. As an additional consequence, the network formation processes will tend to eventually stagnate and become

[8] A myopically optimal move can also involve connecting to other nodes in the network with particularly high synergy or low linking costs. See Section B for additional discussion of heterogeneous costs and benefits of connections.

[9] Kondor et al. (2014)

[10] House et al. (2015)

[11] Eiron and McCurley (2003) and Albert-László et al. (2000) although the evidence is somewhat indirect based on current topology rather than observed dynamics.

[12] Newman (2004)

[13] Wang et al. (2008)

[14] For evidence on history dependence see Puffert (2002) and Puffert (2004)

[15] See Mislove et al. (2013) for an example.

more myopic. Some form of external force, such as technological change or death of old nodes, is needed to maintain rich dynamism and strategically interesting behavior.

In Section 2.3 we present our dynamic network formation model which formalizes the intuition of the example. Our model is characterized by a growing set of nodes, history dependence, and forward looking strategic agents. A growing set of nodes is desirable because the set of agents is not stable in many interesting networks.[16] New firms and consumers are always joining the market; the set of connection opportunities for entrants is changing over time. It is also realistic to assume that in many real-world contexts connections are relatively static. Creating new ties or destroying old ones can be costly. Manufacturers do not immediately change their suppliers in response to small shifts in demand. Our model captures these interesting dynamic features which do not fit well with previous models.

Our combination of strategic agents and history dependence allows for meaningful dynamics. Without history dependence, forward looking behavior becomes irrelevant. In addition, this combination of features characterizes many real -world economic networks. Whereas in previous network formation models, small changes to player behavior only tend to influence which players end up in each position in the finished network, in this model early play can dramatically influence the entire network structure.

Our baseline model includes a requirement that players must connect to one of the dominant nodes as they join the network. This restriction serves three purposes. First, as we show in Proposition 1, the restriction isolates two particularly interesting and important types of behavior: myopic actions and vying for dominance. These types of behavior do arise and are important in more general settings, as we show in the latter portion of the paper, but the possibility of other types of behavior can complicate the discussion. Second, this restriction dramatically improves tractability of the model. We can characterize equilibria in very large networks for the restricted game, while in more general settings it becomes computationally infeasible to characterize equilibria in even modestly sized 15-node networks.

Third, the restriction can provide additional realism in certain environments. In many settings, the most central individual in a network will have some level of gatekeeping authority, either officially, by tradition, or through interaction with an outside agency. For example, major animation studios may require that any new software they use be compatible with the dominant software to facilitate workflow. As such any software which is not compatible with the most central nodes would be at a severe disadvantage. It should be noted that in small networks such as the example in Section 2.3, players will naturally choose moves which satisfy the restriction without being required to do so. At the end of Section 2 we also provide an example which demonstrates how the phenomenon of vying for dominance can arise.

The main results of the paper are presented in Section 3. The first result states that in all Markov Perfect Equilibria all players will choose either to move pseudo-myopically[17] or vie for dominance, with no other moves ever being optimal. This generality implies that, at least in the restricted game, the distribution of profitable vying opportunities is what determines node centrality.

The second result in Section 3 shows that the frequency of opportunities to vie for dominance decreases exponentially as the game progresses even when indifferences are resolved in a manner that favors newer nodes. The most dominant nodes in the resulting networks generally entered the network relatively early. A more general version of this result, in which players break ties in favor of the newer nodes with a fixed probability, is provided in Appendix D.

In Section 4 we discuss a less restricted version of the game. We rework our four-node example, showing that vying for dominance can still play a critical role in determining which nodes become central even in a less restricted setting where players are not required to connect to dominant nodes. We also characterize equilibrium behavior when the cost of connections is very high or very low, although equilibrium behavior with intermediate connection costs straightforward. The final result in Section 4 demonstrates that players will tend towards making myopic, single connection moves near the end of the network formation process. This result confirms that the process tends toward stagnation, with early players having more opportunities for profitable strategic behavior like vying for dominance.

Section 5 discusses the efficiency of outcome networks and options for enforcing efficient results. There is a precise cost threshold which determines efficiency. Below that threshold the complete

[16] For an example see Mislove et al. (2013)

[17] More precisely they may choose from a small set of moves which includes myopic action.

network is the only efficient outcome, while above the threshold, only the star network is efficient. Contrary to previous results in static network formation models,[18] in our setting, it is possible for both inefficiently over-connected and inefficiently under-connected networks to form in equilibrium. Intuitively, under-connection can occur in most network formation models because connections can provide positive externalities, in the form of additional centrality, to players other than the connection sponsor. Over-connection can occur in our model because strategic behavior like vying for dominance can generate negative externalities for other vying players through competition for future connections.

Regardless of the parameter values involved, the fact that we are working with a finite game of perfect information makes it relatively easy to enforce efficient outcomes using a centralized mechanism. As a final point in Section 5, we present a novel decentralized mechanism which can guarantee an efficient outcome in any finite game of perfect information.

Appendix B examines a number of further modifications and extensions to the model. We address what happens to the results in the base game when we allow players to make zero connections, when we allow for heterogeneity in connection benefits, and when we allow players to make additional connections after they first connect to the network.

While the logic in these extensions is more complex than in the base game, the main points of our paper persist across these formulations. First, the tendency of later players to connect to central nodes gives early players an incentive to make more connections than they otherwise might, potentially enticing players to vie for dominance. Second, the cost of becoming highly central increases over time, and benefits of high centrality are limited. This leads to early mover advantage and eventual stagnation where new dominant nodes do not arise.

1.1 Literature on Network Formation

There has been a great deal of work on network formation in the past, but previous models did incorporate strong, irreducible strategic dynamics. These models either lacked dynamics entirely, used agents that were not fully forward looking, or used restricted setups in which the set of possible solutions depends only on static features of the networks, such as stability or efficiency.

Economic models of network formation have traditionally been stability-based with few dynamic features. For example Jackson and Wolinsky (1996) propose a model of cooperative network formation[19] in which a network is stable if and only if every player who is part of a connection wants to keep that connection and no two players who are not connected want to connect. Note that this stability concept is cooperative because players need to agree to make connections.

Our model comes from a different area of the network formation literature developed by Bala and Goyal (2000). In their non-cooperative models of network formation, a network is stable if and only if every person who is sponsoring a connection wants to maintain that connection and no player wants to sponsor a new connection.[20] This model is non-cooperative in the sense that players can make connections unilaterally. Our model takes its general structure from Bala and Goyal (2000) and adds strategic dynamics. It should be noted that Bala and Goyal (2000) did discuss a dynamic version of their model, but they did not allow for forward looking strategic agents.

Special attention should be given to how results from the two-way flow with decay model of Bala and Goyal (2000) compare to those in our paper, because that is the closest version of their model to the one we employ. In Bala and Goyal (2000)'s model non-degenerate networks can form, but our model makes it much easier to produce non-degenerate networks even at high cost levels, because the dynamic nature of the game can provide large benefits for multiple-connection moves.

As a consequence, our model permits inefficient over-connection of a type that is not possible in the Bala and Goyal (2000) model. In particular, it is possible to generate networks which include edges that it would be overall welfare improving to remove. In the Bala and Goyal (2000) model, this is impossible, because connections only produce positive externalities. If removing a connection would improve welfare, it would also be guaranteed to improve the payoff of the sponsoring player, who would therefore terminate the connection. In our model, however, vying can introduce inter-temporal

[18]See Bala and Goyal (2000)

[19]For a comprehensive look at this type of network see the book by Jackson (2008)

[20]For a comprehensive examination of this type of network see the book Goyal (2007)

negative externalities. By vying, a player can decrease the payoffs of other vying players. This allows inefficient over-connection to arise.

As an additional note, the inherent asymmetry created by the order of entry tends to produce less regular networks than those considered in Bala and Goyal (2000). For example, achieving a ring network, in which all nodes are connected to exactly two neighbors, is simple in the Bala and Goyal (2000) framework but is effectively impossible in ours. To form a ring in our model, the last player would need to connect to two end nodes on a chain. The last player to join the network has strong incentives not to connect to the ends of a chain but instead to connect to more central nodes closer to the middle.

The concept of farsighted stability into the domain of cooperative network formation has been explored by Page et al. (2003), Dutta et al. (2005), and Herings et al. (2009). This work does, to some extent, introduce a form of implied dynamics and simple foresight, but the models do not have a formal dynamic structure, and the agents are not forward looking in the subgame perfect sense.

Many recent models include dynamics in a more direct manner.[21] However, the agents in these models are not forward looking. In Watts (2001), for example, players are assumed to update their connections myopically without regard to future consequences. In papers like Kim and Jo (2009) and Duernecker and Vega-Redondo (2017) players only receive payment as they are joining the network, so future periods are irrelevant to the current mover.

Several models include both non-trivial dynamics and forward looking strategic agents.[22] However, these models usually employ special setups in which the feasibility of achieving a particular network depends only on static features of that network. For example, Song and van der Schaar (2015) propose a model with a repeated game-like structure, so all networks which produce more than min-max payoffs for all players are feasible solutions. In Mutuswami and Winter (2002) and Currarini and Morelli (2000) all efficient networks can be formed using centralized mechanisms.

There are a few dynamic network formation models in which solutions do not depend on static features of the networks, although additional simplifying assumptions are generally used. In the model of Aumann and Myerson (1988), the payoff function guarantees that only complete connected components can form. In other words, all nodes in a "group" must be connected to all other nodes in that group. Only the number of nodes in a particular group matters, because only one structure is possible for a given group size. This allows the network formation model to be reduced to a more standard model of dynamic coalition formation.

The model of Chowdhury (2008) is one of the most similar to our own. Both models include sequential link formation and forward-looking strategic agents. In addition, there is the possibility in Chowdhury (2008) for early movers to make myopically sub-optimal moves in hopes of gaining future connections, which can be thought of as loosely similar to the "vying for dominance behavior" of our model. However, Chowdhury (2008) assumes that each node can only sponsor one connection, and thus rules out by assumption the possibility that nodes may compete for centrality by making multiple connections.

2 Model

We now present the basic network growth model. This model, as well as others discussed later in this paper, are similar to the game presented in the accompanying experimental paper Neligh (2019). This ourodel is effectively a dynamic version of the two way-flow with decay model of Bala and Goyal (2000) with the addition of a connection restriction detailed below.

In s model, there are J total players, represented by J nodes. New nodes are added to the network individually over time. As each node is added, the corresponding player chooses some group of existing nodes to connect to. The set of chosen connections must be non-empty. The game concludes, and payoffs are awarded, after the last player joins the network and makes their connections. Players benefit from centrality, but they must pay a cost for the connections they have made.

[21] Such as Watts (2001); Kim and Jo (2009); and Vazquez (2003)
[22] See Mutuswami and Winter (2002); Currarini and Morelli (2000); and Song and van der Schaar (2015)

We now present the model more formally. Lower case letters refer to indices and non-parameter scalar values. Model parameters and networks are referred to by upper case letters. Edges between two nodes are referred to using tuples of the form (index 1, index 2). Bold lower case letters refer to sets of nodes, sets of edges, and vectors.

The set of players and corresponding nodes is indexed $j \in \{1, ..., J\}$. Networks $G = \{\mathbf{n}(G); \mathbf{x}(G)\}$ are composed of a set of nodes $\mathbf{n}(G)$ and a set of edges $\mathbf{x}(G)$. Edges are represented by pairs of nodes. Ordering of nodes in the pair is irrelevant, because connections are not directed. There are J time periods indexed $t \in \{1, 2, ...J\}$. G_t refers to the network formed at the end of period t. Every period one node representing one player joins the network, so player, node, and time indices are generally interchangeable. The game starts with the network $G_1 = \{1; \emptyset\}$, containing Node 1 and no connections.

Before we discuss player actions and strategies we need a definition.

Definition. *The set of dominant nodes $\mathbf{d}(G_t)$ is defined by*

$$\mathbf{d}(G_t) = \left\{ \arg\max_{i \in G_t} \sum_{j \neq i} \delta^{d_{ij}(G_J)-1} \right\}$$

In other words, it is the set of nodes with the highest centrality.

Strategies are mapping from every state a Player j could encounter, G_{j-1}, to a distribution over sets of connections $\mathbf{h}_j$. Note that there is a one to one mapping between action histories and networks, so our definition for strategy just a contextualization of the standard definition of strategies in extensive form games with perfect information. We restrict $\mathbf{h}_t$ to only contain connections linking Node j with an existing Node $i < j$.

We also require that $\mathbf{h}_t$ contains at least one element of $\mathbf{d}(G_{t-1})$. In other words, players must connect to a dominant node as they join the network. We refer to this requirement as the Dominant Node Restriction or DNR. It should be noted that the restriction is meaningful in the sense that there is no guarantee that the equilibrium of the restricted game is also an equilibrium of the game without this restriction. As such, equilibrium refinements cannot be used to achieve the same effect as this restriction.

After Player t chooses their connections, the network is updated using the relationship:

$$G_t = G_{t-1} \cup \{t; \mathbf{h}_t\}$$

The updated network, which will serve as the basis for the next player's move, is simply the old network plus the new node and the connections chosen by the new player.

After Player J moves, the final network G_J is generated, and the game ends.

Each player then receives their payoff based on the utility function:

$$u_i(\mathbf{h}_i, G_J) = \zeta_i(G_J, \delta) - C|\mathbf{h}_i|$$

$C|\mathbf{h}_i|$ is the cost of connections made by individual i. It is $C \in \mathbb{R}^+$, a constant cost of connections, times the magnitude of the set of connections made by i, $\mathbf{h}_i$. $C \in \mathbb{R}^+$ is the constant cost of connections. $\zeta(G_J, \delta) = \sum_{j \neq i} \delta^{d_{ij}(G_J)-1}$ is a standard measure of closeness centrality. Decomposing $\sum_{j \neq i} \delta^{d_{ij}(G_J)-1}, \delta \in (0, 1)$ is a geometric discount factor. $d_{ij}(G_t)$ is the minimum distance between Node i and Node j in edges under network G_t. The minus one in the exponent adjusts the term such that we do not have to normalize C with respect to δ in order to guarantee that a direct connection has a benefit of one.

This type of payoff function is fairly standard and is used in Bala and Goyal (2000), Watts (2001), and Jackson and Wolinsky (1996). This type of network payoff is particularly useful for describing systems where benefits spread from their source nodes at a steady rate, providing value that decays over time. However, this type of payoff can also approximate any system where more central nodes gain more benefits, as this measure of centrality is highly correlated with other measures of centrality, especially in networks with low diameter. Valente et al. (2008) examine the correlation between different types of centrality in real-world networks.

2.1 Preliminary Definitions

Having set up the game, we now take the opportunity to formally define several terms.

Definition. *A myopic move for Player i is an element of*

$$\arg\max_{\boldsymbol{h}_i} u_i(\mathbf{h}_i, G_i)$$

In other words it is a move that would be optimal if the game ended immediately.

Definition. *"Vying for dominance" is any move $\boldsymbol{h}_i$ by Player i such that*

$$i \in \arg\max_k \sum_{j \neq i} \delta^{d_{ij}(G_J)-1}$$

In other words it is a move which results in the player being one of the dominant nodes in the network immediately after.

These definitions will be referenced through the remainder of the analysis.

2.2 Solution Concepts

In this paper we will be using two solution concepts. The solution type is being used in each instance will be specified in formal statements. The first solution type of interest is Subgame Perfect Equilibrium (SPE), because we want to be able to capture strategically dynamic behaviors of forward looking agents. We use the standard definition of subgame perfect equilibrium: a strategy profile in which every action chosen with some positive probability at a particular information set is optimal for the subgame beginning with that information set. Because this is a finite game of perfect information, there is a one to one correspondence between information sets and action histories, so existence of a SPE is assured.

Due to the natural symmetries of networks, indifferences are quite common in this game. This feature can potentially allow for sunspot-like behavior, where players change the way they resolve indifferences in response to payoff-irrelevant changes in the action history. These sunspots can lead to very poorly behaved equilibria, so in settings where this is an issue, we use the more refined Markov Perfect Equilibrium (MPE). In very loose terms, this means that players are assumed to choose the same move in similar situations, where situations are considered similar if the same action is optimal and the set of available actions is the same.

More formally stated, Markov Perfect Equilibrium (MPE) requires that each player in equilibrium conditions their moves on the "coarsest" possible partition of game histories which is not coarser than the partition which defines the set of available moves.[23] In our game we do not have to consider the precise definition of "coarseness" because it is actually possible to construct an equilibrium where each move depends only the set of available moves, which is automatically the coarsest partition which can be used.

Note, in the case of both SPEs and MPEs, solutions are often not unique. Many networks have a symmetric structure which can lead to multiple sets of connections behaving essentially identically from a payoff standpoint. The resulting indifferences between moves frequently lead to a multiplicity of equilibria. Because the multiplicity arises due to structural indifferences rather than beliefs, we cannot eliminate or reduce the multiplicity through the use of standard equilibrium refinements. Therefore, in addition to a set of parameters, we also need a tie-breaking rule to generate a unique equilibrium.

Definition. *Tie-Breaking Rule: a tie-breaking rule refers to some rule by which players resolve indifferences in the construction of a SPE.*

A player's tie-breaking rule is a possibly stochastic, possibly history dependent way of completing the player's partial preference order over moves during the backwards induction process. It should be noted that most intuitive consistent tie-breaking approaches, such as always favoring connections with newer nodes, will generate MPEs.

[23] For the full technical definition see Maskin and Tirole (2001).

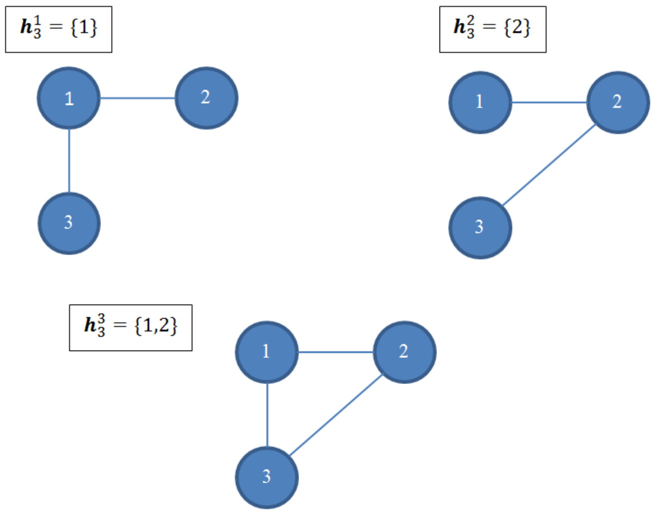

Figure 2: Immediate outcomes of the moves of Player 3

2.3 Example

In order to demonstrate what an equilibrium of the game can look like and how tie-breaking rules play a role, we present an example with four nodes. Nodes 1 and 2 have no decisions to make, so we shall ignore them. Assume $C > (1-\delta)$.[24] We solve for the SPE through backwards induction.

Player 4: Given that $C > (1-\delta)$, Player 4 will always make exactly one connection to one of the dominant nodes. By assumption, he must connect to one dominant node.

Connecting to one of the most central nodes always results in Player 4 being a distance of two from the nodes he did not connect to. Each additional connection would increase his cost by C while providing a benefit of at most $(1-\delta)$, the benefit of decreasing the distance to another node from two to one. If there are multiple dominant nodes, then Player 4 will be indifferent as to which node he connects to, so his move will be determined by the tie-breaking approach.

The fact that additional connections provide a myopic benefit of $(1-\delta)$ is a general feature of the game resulting from the fact that all nodes in the network are always directly connected to all dominant nodes. This can be shown through a simple induction. If all nodes are currently connected to the most connected nodes, and the current player must connect to at least one of the most connected nodes, then at the end of the current turn, all players will still be connected to the most connected nodes. Thus, players will always either be a distance of two from each other or directly connected.

Player 3: Player 3 has three options for $\mathbf{h}_3$, $\mathbf{h}_3^1 = \{1\}, \mathbf{h}_3^2 = \{2\}, \mathbf{h}_3^3 = \{1,2\}$. Note that both Node 1 and Node 2 are dominant in the network Player 3 faces, so he has no restriction on which connections he can make beyond having to make at least one. Each move leads to a different G_3 as illustrated in Figure 2. Note that $\mathbf{h}_3^1$ and $\mathbf{h}_3^2$ are both myopic moves while $\mathbf{h}_3^3$ is vying for dominance.

After $\mathbf{h}_3^1$, the only possible result would be G_4^1, since there is only one element of $\mathbf{d}(G_3)$. Player 4 will connect to the unique element $\mathbf{d}(G_3)$ (see Figure 3). We will ignore $\mathbf{h}_3^2$ since it is symmetric to $\mathbf{h}_3^1$.

Depending on Player 4's tie-breaking rule, move $\mathbf{h}_3^3$ could eventually lead to G_4^3 , G_4^4, or G_4^5 (see Figure 4). This means that Player 3's expected utility from this move depends on the parameters of the game and his beliefs about Player 4's tie-breaking rule.

If Player 3 believes that choosing $\mathbf{h}_3^3$ will lead to G_4^3 or G_4^4, Player 3 will not choose that move, because $\mathbf{h}_3^3$ costs C more than $\mathbf{h}_3^1$ or $\mathbf{h}_3^2$ and only provides an additional benefit of $(1-\delta)$.

If instead Player 3 believes that $\mathbf{h}_3^3$ might lead to G_3^5, the decision is slightly more complicated. Now the benefit of $\mathbf{h}_3^3$ could be as high as $2(1-\delta)$ which could be enough to compensate for the

[24] The case where $C < (1-\delta)$ is discussed in the proof of Proposition 1.

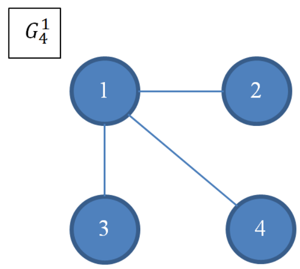

Figure 3: Final result of player 3 choosing move $\mathbf{h}_3^1$.

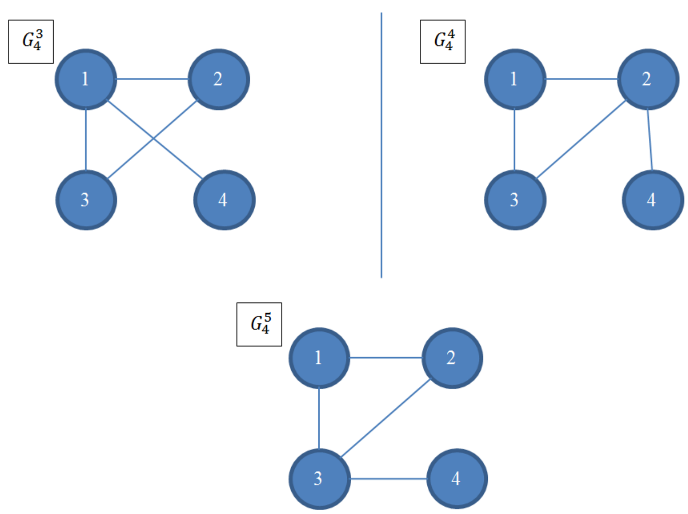

Figure 4: Possible final outcomes if player 3 chooses move $\mathbf{h}_3^3$.

additional cost. If move $\mathbf{h}_3^3$ leads to outcome G_3^5 with a probability of $P(G_3^5)$, then Player 3 will choose $\mathbf{h}_3^3$ as long as $(1-\delta)(1+P(G_3^5)) > C$. It is the expectation of future connections that gives value to vying for dominance.

In this example, only those who have many established connections can potentially receive a connection from Player 4. Player 4's behavior can be considered as a form of preferential attachment, which has important strategic ramifications.[25]

To some extent, the existence of vying for dominance behavior can be thought of as related to a phenomenon found by of Akerlof and Holden (2016) where players bid early on for a resource which allows them to become the center of a network that is formed later, but in our setting the investment itself is part of the network formation process. Players can invest in connections early on in order to receive more connections later.

Vying for dominance occurs even when players are not required to connect to dominant nodes. As we discuss in Section 4.1, and Appendix B, myopic behavior will frequently Player 4's behavior in the example, incentivizing vying for dominance even without connection restrictions. Behavior in these more general models can be complicated by other considerations, however.

3 Main Results

In this section we present the two main results of the paper. The first result demonstrates the importance of vying for dominance in determining which nodes become central during the network formation process, while the second result elucidates an important mechanism behind early mover advantage in network formation.

The SPEs of the game are not necessarily well behaved, so for the first two results we will focus on MPEs. The importance of using MPE to the first result in particular is discussed in more detail in Appendix C.2.

3.1 Behavior in Markov Perfect Equilibria

The first major result demonstrates how equilibrium behavior can be exhaustively characterized as myopic or vying, providing a clean separation between different types of play and substantially simplifying further equilibrium characterization.

Proposition 1. *If $C \neq 1-\delta$ then, in all Markov Perfect Equilibria of our baseline network formation game, $\mathbf{h}(G_j) \subseteq \mathbf{d}(G_j)$ (a pseudo-myopic move) or $\mathbf{h}_j(G_j) = \{1, 2, ..., j-1\}$ (vying for dominance) $\forall j$.*

In other words every node will either only connect to a subset of the dominant nodes (playing pseudo-myopically) or connect to every node in the network (vying for dominance). We say pseudo-myopically here, because while players will generally connect to a single dominant node, which is truly myopically optimal, there are rare situations where connecting to multiple dominant nodes is optimal. See Appendix C.3 for an example.

Proof of Proposition 1: The case where $C < 1-\delta$ is simplest, so we cover it first. We work by backwards induction. Player J will always connect to every other player. To see this note that if J is not connected to every player, adding a new connection will increase his utility by $(1-\delta) - C > 0$.

If all future players will connect to every available player, so will player $J-k$. Suppose not, then at the end of the game he will be a distance one from all nodes he connects to (and all nodes that come after him) and a distance of two from the nodes he doesn't connect to. This means that player $J-k$ can always increase his utility by connecting to an additional node if more connections are possible, since the gain, $(1-\delta)$, is greater than the cost, C. Therefore player $J-k$ will connect to all available nodes when he joins the network.

By induction, all players will connect to all available nodes when they join the network. This leads to the formation of a complete network.

[25] Because only the most connected nodes can receive a connection from Player 4, the effect is more exaggerated than what is seen in traditional preferential attachment models Yule (1925).

Next consider the case where $C > 1 - \delta$. We begin by showing that Player J must always choose a pseudo-myopic move, and that his choice among pseudo-myopic moves depends on the network only through the set of dominant nodes. We then show that if all future players choose either a pseudo-myopic move or vie for dominance, and that the choices among these moves do not depend on the current network except through the current set of dominant nodes, then it is optimal for the current node to do the same (and no other moves are optimal). By induction this will give us the result.

Note that there is a one to one correspondence between the set of dominant nodes and the set of possible actions, so conditioning on one is the same as conditioning on the other.

Player J: We want to show that Player J will always only connect to $b_{J-1} \in \mathbf{d}(G_{J-1})$ (pseudo-myopic move). Player J will always connect to some b_{J-1}, by the DNR. Any additional connection can only provide Player J benefit of $1 - \delta$. The marginal utility of each additional connection would be $1 - \delta - C < 0$. Therefore, all other possible moves are strictly worse than $\mathbf{h}_J = b_{J-1} \in \mathbf{d}(G_{J-1})$ (pseudo-myopic move).

Player J is indifferent between all singleton pseudo-myopic moves, as such, by Markov Perfection, he can only condition his move on the set of dominant nodes $\mathbf{d}(G_{J-1})$.

Player $J - k$: In examining the move of Player $J - k$'s move, we will assume the following induction hypothesis.

Induction Hypothesis 1: All future players play a pseudo-myopic move or vie for dominance, and their behavior depends only on the set $\mathbf{d}(G_{t-1})$.

Note that in this induction hypothesis, we are not requiring that the sequence of future moves include both types of move. If the future sequence of moves must include only pseudo-myopic moves, the induction hypothesis is still satisfied.

If Player J is the only remaining player then this induction hypothesis is satisfied, since we have already shown that Player J's behavior meets this criterion. If we can show that under Induction Hypothesis 1, Player $J - k$ will also only choose a pseudo-myopic move or vie for dominance, and his choice will only depend on the network state through $\mathbf{d}(G_{J-k-1})$, then we are done.

We prove this using two lemmas.

Lemma 1. *Under Induction Hypothesis 1, Player $J - k$ will always choose a pseudo-myopic move or vie for dominance.*

Proof of Lemma 1: The lemma follows from the fact that the utility that Player $J - k$ receives depends only on the myopic utility of his action (direct costs and benefits from making current connections), the behavior of future players (gain from future connections), and the fact that these components are additively separable. Since the probability of any set of future actions depends only on $\mathbf{d}(G_{J-k})$, any optimal action by Player $J - k$ must be the myopically optimal way of achieving the resulting $\mathbf{d}(G_{J-k})$.

Every dominant node that Player $J - k$ connects to is going to remain a dominant node. No other node that Player $J - k$ connects to will become a dominant node. Player $J - k$ can only become a dominant node by connecting to all players, since the current dominant node is connected to all players. Since additional connections are myopically harmful, Player $J - k$ wants to choose the move with the fewest possible connections to achieve a given $\mathbf{d}(G_{J-k})$. That means Player $J - k$ will connect to all nodes (vying for dominance) if he wants to add his own node to $\mathbf{d}(G_{J-k})$ or just another a subset of the current dominant nodes (a pseudo-myopic move), if he wants that subset to be in $\mathbf{d}(G_{J-k})$. No other moves can achieve different $\mathbf{d}(G_{J-k})$ or the same $\mathbf{d}(G_{J-k})$ with fewer connections. $\square$

Lemma 2. *Under Induction Hypothesis 1 the expected utility and feasibility of the pseudo-myopic move and vying for dominance depends only on $\mathbf{d}(G_{J-k-1})$.*

Proof of Lemma 2: To begin, note that the feasibility of each pseudo-myopic move and vying for dominance depends only on $\mathbf{d}(G_{J-k-1})$. The set of feasible pseudo-myopic moves is the set of subsets of $\mathbf{d}(G_{J-k-1})$ and vying for dominance is always possible.

The expected value of each of these moves can again be decomposed into myopic and future portions. The myopic utilities from each move are fixed since they depend only on the number of connections made by the player. Players receive a benefit of 1 and pay a cost C for every player they connecte to and receive a benefit of δ for each other player in the network.

The future benefits from all pseudo-myopic moves are similarly fixed given Induction Hypothesis 1, since each pseudo-myopic move will always produce the same $\mathbf{d}(G_{J-k})$ equal to the set of nodes Player $J-k$ connected to. Note that whether or not a given move is pseudo-myopic depends only on $\mathbf{d}(G_{J-k})$ by definition. The future benefits of vying for dominance depend only on $\mathbf{d}(G_{J-k-1})$ since after vying for dominance $\mathbf{d}(G_{J-k}) = \mathbf{d}(G_{J-k-1}) \cap J-k$. $\square$

Given Lemmas 1 and 2, Player $J-k$ will only choose pseudo-myopic moves or vie for dominance, and his choice among these actions only depends on $\mathbf{d}(G_{J-k-1})$. Thus by induction, there exists a SPE of the game where every player satisfies Induction Hypothesis 1.

In the resulting SPE, every player conditions his move only on the set of available moves, so the resulting SPE must be a MPE. Furthermore, any SPE where players do not condition their moves only on the set of available moves cannot be a MPE, since a coarser partition equilibrium exists. $\square$

Note that if $C = 1-\delta$ then then the decision by all players to randomize among all feasible moves constitutes a Markov Perfect Equilibrium. Such an equilibrium generates all feasible networks with positive probability.

Proposition 1 is very strong in the sense that it eliminates a tremendous number of possible moves. Player j who is facing a network with k dominant nodes will have $k2^{j-2}$ possible moves. Proposition 1 eliminates all but 2^k of those moves.[26]

Consider an example in which Player 5 is moving and all existing nodes are connected only to Node 1. Node 1 is the only dominant node in this case, so Player 5 can connect to any combination of nodes as long as he connects to Node 1. In this case there are seven such combinations, but due to Proposition 1, Player 5 will only ever choose one of two moves: he will connect to Node 1 only, or he will connect to all existing nodes. The reduction in the set of potential moves can be even greater for later nodes.

Beyond the ability to reduce the state space of the game, Proposition 1 essentially guarantees that "the rich get richer" in the sense that only the most connected nodes can ever become relatively more connected than other nodes. As such, only vying players can come to dominate the network.

3.2 Exponential Slowdown

How players resolve indifferences can still have a substantial impact on the network that forms even within the class of Markov Perfect tie-breaking strategies. For example, if all players break indifferences in favor of connecting to older nodes (a Markov Perfect tie-breaking rule), the only possible network that can form is the star network centered on Node 1.[27] The dynamics of this network are not particularly interesting and first move advantage arises in a very obvious way.

However, opportunities to profitably vie for dominance occur more often early in the network formation process even when tie-breaking favors new nodes vying.

Definition. *Novelty Seeking Tie-Breaking (NSTB) is a tie-breaking rule which states that, when a player is indifferent between pseudo-myopic moves, he will always choose the move with the lowest total age. Total age of a move is the sum of the ages of all nodes connected to during that move.*

Given this tie-breaking approach, the number of myopic moves between vying players follows a predictable pattern.

Proposition 2. *Under NSTB, if $1-\delta < C$ the unique solution to the baseline network formation game is characterized as follows*

1. *All players will either connect to all existing nodes (vie for dominance) or make a single connection to the newest dominant node (choose a myopic move).*

[26] Unless $k = j-1$ in which case it does not eliminate any moves.

[27] Note that this is not the case in the base game. See C.1 for the counter-example.

2. *If n players vie for dominance during the game, the number of myopic moves between the i^{th} vying player and the $i+1^{st}$ is approximately*

$$\lambda^{n-i}(J-1)\left(1-\tfrac{1-\delta}{C}\right)$$

   Where $\lambda = \frac{(1-\delta)}{C} < 1$.

3. *The difference between the approximation and the true number is less than $n-i+1$.*

This proposition tells us that the time between vying opportunities grows approximately exponentially as the game progresses, and that the quality of the approximation is good, at least when considering later vying players.

Note that this Proposition is actually a special case of Proposition 2* which is presented and proved in Appendix D. As such we will be pointing to parts of that proof for some of the more technical parts aspects. We will prove Part 2 of the proposition in the main text, because this proof provides important intuition regarding early mover advantage in our network formation game.

Proof of Proposition 2: Part 1 of the proposition is immediate from the following Lemma:

Lemma 3. *Under NSTB tie-breaking all players will vie for dominance or connect to the newest dominant node, and the move they pick does not depend on the current network state.*

Lemma 3 is a special case of Lemma 6. See Appendix D for proof of Lemma 6.

Part 2: Having characterized the equilibrium in general terms, we now determine which players will choose the special myopic move and which will vie for dominance.

Consider J sufficiently large as to ensure that all the node indices referenced are positive.

As before we work through backwards induction.

Player J: Begin by considering the move of Player J. His expected utility from the special myopic move is $1+\delta(J-2)-C$, because he will be directly connected to a single node in $\mathbf{d}(G_{J-1})$ and a distance two from all other nodes. As discussed previously, Player J will never vie as long as $C > 1-\delta$.

Player $J-k$: We now consider Player $J-k$ under the following induction hypothesis.

Induction Hypothesis 2: All following players will make a single connection to the newest dominant node.

By Lemma 3, Player $J-k$ will not influence the choice of future players. If Player $J-k$ vies, Player $J-k+1$ will directly connect to Node $J-k$ since Node $J-k$ will be the newest node in $\mathbf{d}(G_J-k)$. If Player $J-k+1$ connects to Node $J-k$, all nodes after $J-k+1$ will connect to node $J-k$, because by then $J-k$ will be the only node in $\mathbf{d}(G_t)$.

Player $J-k$ will receive $1+\delta(J-2)-C$ from playing myopically and $(J-1)-(J-k-1)C$ from vying. In both cases the first term (of the form ...C) represents the costs of connections. The first terms describe the expected benefit from connections.

Player $J-k$ will then choose to vie if

$$(1-\delta)(J-2) > (J-k)C$$

If this relation is not satisfied when $k = J-3$, then no nodes will vie, and the resulting network will be a star on Node 2. We call the first node that satisfies the condition for vying optimality v_1. Call the set of all optimal vying nodes, $\mathbf{v}$.

Player $v_j - k$: Consider now a Player $v_j - k$ who moves k moves before the next vying node, v_j. For now, we assume that player v_j+1 will not also be vying for dominance. We will revisit this assumption later in the proof.

This time we will write the choice in terms of the gains and costs of vying relative to myopic play to reduce extraneous terms. The extra myopic cost (net gains from the immediate connections) is:

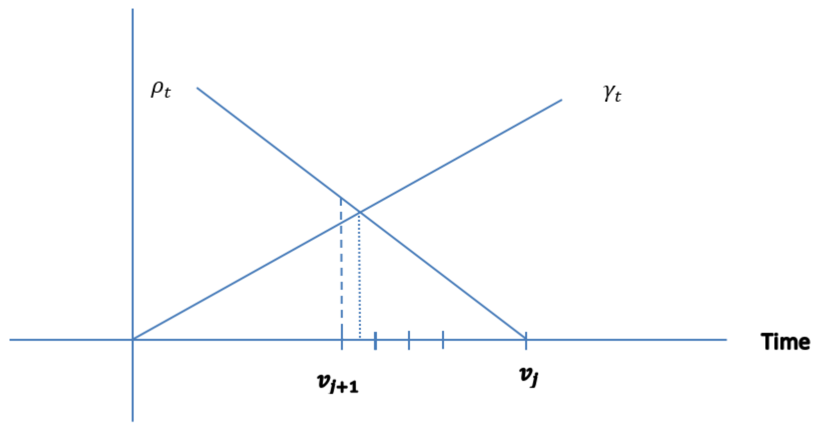

Figure 5: This figure demonstrates how, given v_i, we may find v_{i+1} by going backwards in time until we find a node where optimality is satisfied.

$\gamma_{v_j-k} = (v_j - k - 2)(C - 1 + \delta)$. The relative future benefit of vying, which comes from future direct connections, will be $\xi_{v_j-k} = (k-1)(1-\delta)$. This expression comes from the fact that, after vying, Player $v_j - k$ will receive a connection from the next myopic player and become the only dominant node until the next node vies.

Player $v_j - k$ will vie if the gains are greater than the costs. As the time until the vying player increases and ξ_{v_j-k} increases, and as we move backwards towards the first move, γ_i decreases. This means that, given a fixed next vying player, Player v_j, we can find the previous vying player by going backwards until we find a node such that $\gamma_i < \xi_i$.

Define k_j as the lowest value of k such that $\gamma_{v_j-k_j} < \xi_{v_j-k_j}$, and then say $v_j - k_j = v_{j+1}$. See Figure 5 for a visualization of this relationship.

Now we ignore the integer constraints on node indices and say that a node will choose to vie when the costs equal the benefits. In other words, we assume v_i satisfies $\gamma_{v_i} = \xi_{v_i} \forall i$. We can see the kind of errors that are introduced by this approximation by comparing the dotted line and the dashed line in Figure 5. The dotted line shows where the relative costs and benefits are equal, while the dashed line gives the actual index of Player v_{i+1}. Note that the difference between the dashed line and the dotted line will always be less than one.

Recall that

$$\xi_{v_{i+1}} = (k_i - 1)(1 - \delta)$$

and note that

$$\gamma_{v_{i+1}} = \gamma_{v_i} - k_i(C - 1 + \delta)$$

since node v_{j+1} will have k_j fewer nodes to connect to when vying.
which we can plug in to get

$$\xi_{v_i} - k_i(C - 1 + \delta) = (k_i - 1)(1 - \delta)$$

which simplifies to

$$\xi_{v_i} = k_i C - (1 - \delta)$$

which also implies

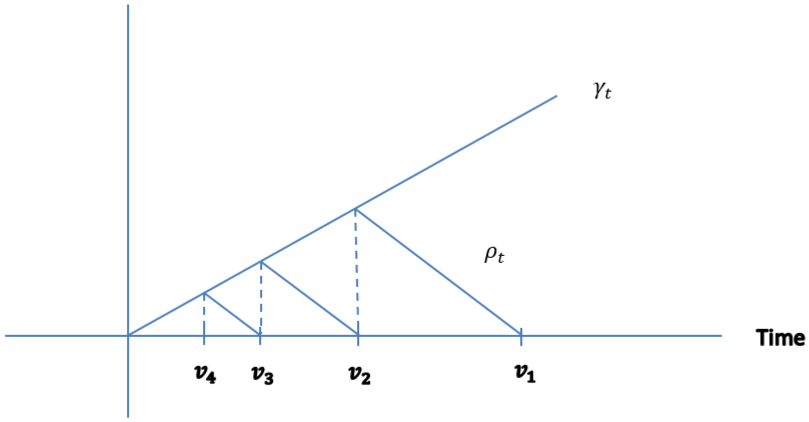

Figure 6: This figure shows how the behavior of the costs and benefits of vying determines when such moves occur.

$$\xi_{v_{i+1}} = k_{i+1}C - (1-\delta)$$

If we substitute these both back in, we get

$$k_{i+1}C - (1-\delta) = (k_i - 1)(1-\delta)$$

which simplifies to

$$k_{i+1} = k_i \tfrac{(1-\delta)}{C}$$

We define

$$\lambda = \tfrac{(1-\delta)}{C}$$

Therefore

$$k_i = \lambda^{j-1} k_1$$

Note that $\lambda < 1$ as long as $C > 1-\delta$, demonstrating an exponential shrinking of the time between vying nodes as we go back in time (hence exponential growth as one moves forward in time).

We can finish the solution by finding and plugging in k_1, which is defined by

$$k_1(1-\delta) = (J - k_1 - 1)(C - 1 + \delta)$$

or

$$k_1 = (J-1)\left(1 - \tfrac{1-\delta}{C}\right)$$

Which gives our solution

$$k_i = \lambda^{i-1}(J-1)\left(1 - \tfrac{1-\delta}{C}\right)$$

Figure 6 shows graphically how this geometric relationship arises.

This means that, if there are n vying players, the length of myopic play between the ith vying player and $i+1$st vying player is

$$\lambda^{n-i}(J-1)\left(1 - \tfrac{1-\delta}{C}\right)$$

Part 3: Recall that Proposition 2 is a special case of Proposition **2***. We do not specifically prove the special case here and instead refer readers to Part 3 of the proof in Appendix D, because the additional notation is cumbersome. The general intuition behind the result is that during the backwards induction process of Part 2, the rounding error can be no larger in magnitude than one for each backwards induction step, so the total error at each step must be less than or equal to the number of backwards induction steps so far. □

This result is generalized to a wider class of tie-breaking approaches in Appendix D. We also argue informally that this stagnation is not merely a product of the finite nature of our game. If the network growth game continued indefinitely with some continuation probability, the maximum possible benefit from vying would be fixed at some finite value. The cost of vying, however, will grow arbitrarily large.

Based on Proposition 2, we can see that vying behavior can become rapidly less common as the game progresses. Towards the end of the game only myopic behavior will occur. Recall that $k_1 = (J-1)\left(1-\frac{1-\delta}{C}\right)$, so if $\frac{1-\delta}{C}$ is small, it is quite possible that only myopic moves will occur for most of the game.

4 Generalized Game

We have shown that vying for dominance plays a critical role in determining which nodes become central and how the increasing cost of vying can give early movers an advantage. It is natural to conclude that both of these results arise from the requirement that players must connect to a dominant node as they join the network. In this section, we will show that vying for dominance and early mover advantage persist even when no such requirement is in place.

To that end, we introduce the less restricted game.

Definition. *The Less Restricted Game (LRG) is identical to the baseline game except that players are not required to connect to a dominant node as they join the network. Instead we only require that they make at least one connection to some existing node as they join.*

In this section, we will show that the main messages of the paper can also be applied to the LRG, although in a less precise way. The same principals can also be applied in some ways to even more general settings, as we show in Appendix B, although in many cases only weak statements can be made.

In this section Markov Perfection does not provide significant benefits, so we focus on general features of SPEs. It is important to note that even with Markov Perfection we would not be able to guarantee that players only choose myopic and vying moves in equilibrium of the LRG. We provide an example in Appendix C.1 where it is optimal for a player to make one connection to a non-dominant node in order to encourage a later player to vie for dominance.

4.1 Example Revisited

Vying and myopic behavior do still remain important in the LRG, as demonstrated by the following rework of the example from Section 2.3. We will show that LRG produces identical behaviors in this small network setting.

As before, Nodes 1 and 2 have no decisions to make, so we shall ignore them. Assume $C > (1-\delta)$.[28] We solve for the SPE through backwards induction.

Player 4: Given that $C > (1-\delta)$, Player 4 will always make exactly one connection to one of the dominant nodes.

Making one connection to one of the most central nodes is always the best single-connection move that the last player can make. This comes from the fact that the centrality of the last player connecting to one node is δ times the centrality of the node he connected to plus one.

Connecting to one of the most central nodes always results in Player 4 being a distance of two from the nodes he did not connect to. Each additional connection would increase his cost by C with a benefit of at most $(1-\delta)$, the benefit of decreasing the distance to another node from two to one.

[28] The case where $C < (1-\delta)$ is covered by Proposition 3.

Connecting to any number of nodes that are not most central is strictly worse than connecting to one of the most central nodes.

If there are multiple dominant nodes, then Player 4 will be indifferent as to which node he connects to, so his move will be determined by the tie-breaking approach.

Player 3: Given that Player 4 displays identical behavior in the base game and LRG versions of this example, Player 3's expected payoffs from feasible moves are also identical across both versions. In addition, the set of feasible moves for Player 3 is still $\mathbf{h}_3^1 = \{1\}, \mathbf{h}_3^2 = \{2\}, \mathbf{h}_3^3 = \{1, 2\}$.

Because Player 4 is naturally incentivized to connect to a dominant node, Player 3 has the incentive to vie for dominance.

4.2 LRG Results

As previously mentioned, it is difficult to establish general rules for equilibrium outcomes in the LRG, but we can provide some useful results for extreme parameter values. These results will also sprovide evidence that early mover advantage is also a common feature of the LRG. Note that all of the results for the LRG also hold for the baseline game, with proofs being essentially identical.

First, we consider conditions for the formation of complete networks.

Proposition 3. *The following rules characterize the formation of complete networks:*

1. *If $C < (1 - \delta)$, then the complete network is the unique network which can form in SPEs of the LRG.*

2. *If $C > (1 - \delta)$, then the complete network cannot be formed by any SPE of the LRG.*

Proof of Proposition 3:

Part 1: We work by backwards induction. Player J will always connect to every other player. To see this note that if J is not connected to every player, adding a new connection will increase his utility by at least $(1 - \delta) - C > 0$.

If all future players will connect to every available player, so will player $J - k$. Suppose not, then at the end of the game he will be a distance one from all nodes he connects to (and all nodes that come after him) and a distance of at least two from the nodes he does not connect to. This means that player $J - k$ can always increase his utility by connecting to an additional node if more connections are possible, since the gain, $(1 - \delta)$, is greater than the cost, C. Therefore player $J - k$ will connect to all available nodes when he joins the network.

By induction, all players will connect to all available nodes when they join the network. This leads to the formation of a complete network.

Part 2: If $C > (1 - \delta)$, then the complete network is no longer a possible outcome of any SPE. To see this consider the move of Player J. This player cannot connect to all other nodes in any SPE of the game. If he was connected to all other nodes but one, then the additional benefit from connecting to the last node would be $(1 - \delta)$. This marginal benefit is less than the marginal cost of C. Since Player J must choose a myopically optimal move in any SPE, he cannot choose to connect to all nodes. □

Next we consider the conditions for the formation of star networks.

Proposition 4. *The following rules characterize the formation of star networks:*

1. *If $C > (J - 1) - \frac{1 - \delta^{J-3}}{1 - \delta}$, then the star networks centered on Node 1 and Node 2 are the only networks which can form in any SPE of the LRG.*

2. *If $C < J - 2$, $J > 3$, and δ is sufficiently small there exists a SPE of the LRG which does not always generate a star network.*

Proof of Proposition 4:

Part 1: The greatest benefit that a player can receive for making one additional connection is the benefit Player 3 gets when, if Player 3 makes one connection, all future players connect in a chain moving away from Player 3, but when Player 3 makes two connections, all future Players connect directly to Player 3. A chain leading away from Node i is the worst possible network for Node i and a star centered on Node i is tied for the best. If Player 3's choice in this case selects between these two outcomes, Player 3's gain from making the extra connection is the difference $(J-1)-\sum_{i=1}^{J-1}\delta^{i-1}$.

Simplifying and rearranging we get $(J-1)-\frac{1-\delta^{J-3}}{1-\delta}$. This is the maximum possible benefit that any player can get from making an additional connection under any strategy profile. We now employ backwards induction to arrive at the result.

Player J will connect to an $i \in \mathbf{d}(G_{J-1})$. To see this, note that once Player J has made one connection, the benefit of each additional connection is always going to be less than the maximum possible benefit from a connection, which is less than C. The best node to make a single connection to in this case is a dominant node by the definition of dominant nodes and the structure of the utility function.

Now consider Player $J-1$. This player will also prefer every outcome where he make only one connection to every outcome where he must make multiple connections. If Player $J-1$ is going to make one connection, it is optimal for him to connect to a dominant node, knowing that Player J will then connect to that same dominant node. If a player connects to one dominant node, that node will be the only dominant node in the next period.

Now consider the move of Player $J-k \geq 3$. By similar logic, Player $J-k$ will always connect only to one node. Knowing that each future player will connect to a dominant node, Player $J-k$ will also connect to a single dominant node.

Player 3 will connect to either Node 2 or Node 1 since both are dominant nodes in G_2. Players 1 and 2 do not have any choices. Therefore all nodes will connect either to Node 1 or Node 2.

Part 2: See Appendix A.2.

4.3 Early Mover Dominance

A Corollary of Proposition 4 demonstrates how opportunities for strategic play will generally be front-loaded in the game. Only early movers will be able to profitably vie for dominance, leading to a tendency for early movers to dominate the final outcome network.

Proposition 5. *If δ is sufficiently small and $C > J-j+1$ then all players after Player j will connect to the same dominant node.*

Proof of Corollary 5: We begin by employing backwards induction with the following induction hypothesis:

Induction Hypothesis 3: All players after Player $J-k > j$ will connect to a single dominant node.

The reasoning showing that under Induction Hypothesis 3 player $J-k$ will always prefer to either make one connection to a dominant node or vie for dominance follows the same pattern as the similar claim in the proof of Part 2 of Proposition 4, so we omit that step here. Here we show that it will never be profitable under Induction Hypothesis 3 for Player $J-k > j$ to vie for dominance when δ is sufficiently small.

The greatest possible net benefit Player $J-k$ could get from vying for dominance is bounded above by the ideal benefit Player $j+1$ could receive by vying.

The ideal vying move in terms of net benefit would involve Player $j+1$ making $N \geq 1$ extra connections and receiving direct connecitons from all future nodes and being a distance two from all existing nods he did not connect with. Assume all nodes would otherwise be an infinite distanc away and provide no benefit. The resulting net benefit is $(J-j-1+N)+(j-N)\delta-NC$. Given that $C > 1$, this net benefit is maximized when $N = 1$, since it is impossible to vie for dominance by sponsoring only connection in any case other than the trivial move of Player 2. However, since we have assumed

$C > J - j + 1$, it must be the case that $(J - j) + (j - 1)\delta - C$ is negative as long as δ is sufficiently small. Therefore Player $J - k > j$ will connect to a single dominant node.

We have shown that all Players after Player j will connect to a single dominant node. To see that they will all connect to the same dominant node it suffices to notice that after Player $j + 1$ connects to a single dominant node, that node will become the sole dominant node, and all later players will connect to it. □

Intuitively, as the game progresses, the maximum future benefits of multiple connection moves decrease, since fewer changeable moves are left. When δ is low and C is high, the maximum net myopic benefits from multiple connection moves are low. Therefore, when all of these conditions hold (few future moves, low δ, high C), multiple connection play is avoided and the network formation process collapses to a predictable myopic pattern.

As such, players arriving late cannot profitably gain dominant positions in the network. This corollary also implies network stagnation. The decreasing benefits and increasing costs of strategic play are causing it to die out.

5 Efficiency

So far we have discussed the types of outcome networks we should see in the LRG and baseline games, but we have not discussed whether these outcomes are efficient. Results on efficiency can provide important information regarding the desirability of the solutions explored in other sections.

When $C > (1 - \delta)$ most of the possible outcome networks are not Pareto ranked, but we can find the most efficient network using a stricter definition.

Definition. *We say an outcome network G_J is efficient if it generates the highest possible sum of utilities out of all feasible outcome networks for given parameters.*

This definition is largely equivalent to the strong efficiency discussed in Jackson and Wolinsky (1996). Note that the LRG and baseline games both have the same payoff functions, and the baseline game does not prevent the formation of any of the networks that are efficient in the LRG, so the results in this section apply equally to both games.

Given this definition we get the following result.

Proposition 6. *Efficient networks are characterized as follows:*

1. *If $C < 2(1 - \delta)$, then the complete network is the unique efficient network.*
2. *If $C > 2(1 - \delta)$, then the star networks (on Node 1 or Node 2) are the only efficient networks.*
3. *If $C = 2(1 - \delta)$, then all feasible networks which contain stars are efficient.*

Proof of Proposition 6: See Appendix A.1.

Note that, because efficiency is not dependent on dynamics, this result is effectively identical to Proposition 5.5 of Bala and Goyal (2000) other than the infeasibility of the empty network in our game.

If we cross reference the efficiency results with the outcome results from Section 4, we can see that inefficient over-connection and inefficient under-connection are both possible.

When costs are low complete networks will form, and when they are high star networks tend to form. When cost levels are intermediate, complete networks cannot form, and star networks are not guaranteed. Propositions 6, 3, and 4 are summarized visually in Figure 7.

5.1 Enforcing Efficiency Using Centralized Mechanisms

The tendency for later moves to be myopic can lead to few connections, which in turn can lead to inefficiency, particularly when connections are welfare-improving. However, achieving efficiency through a centralized mechanism is actually fairly trivial. If $C < 2(1 - \delta)$ then a planner could add a

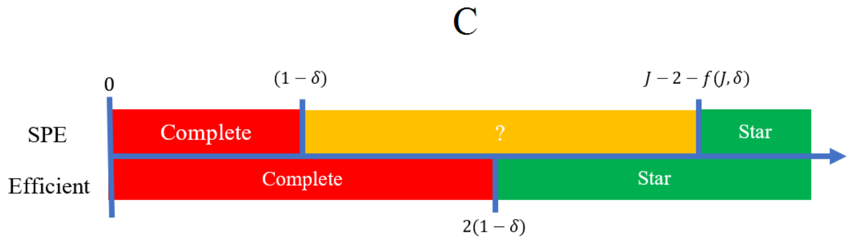

Figure 7: Visualization of parameter regions of interest. Note that the $(J-2-f(J,\delta))$ line can be either to the left or the right of the $2(1-\delta)$ line depending on parameters.$f(J,\delta)=\frac{\delta-\delta^{J-3}}{1-\delta}$.

subsidy of $C-(1-\delta)+\epsilon$ per connection. The effective cost of connections is reduced below $(1-\delta)$ so the complete network forms. Since a total of $J(J-1)/2$ connections will be made in SPE for this effective cost level, the budgetary subsidy could be offset by a total lump sum tax of $(C-(1-\delta))\frac{J(J-1)}{2}$ divided in some way between participants. If the average number of connections made without the subsidy is K, the total welfare gain from such a scheme will always be at least $(2(1-\delta)-C)\left(\frac{J(J-1)}{2}-K\right)$.

If $C>2(1-\delta)$, the planner should instead impose a tax of $(J-1)-C$ per connection. The effective cost of connections would then be increased to $(J-1)$ which is high enough that only the star network will form. A total of $J-1$ connections will be made, leading to a total revenue of $((J-1)-C)(J-1)$. If the average number of connections made without the tax is K, then the efficiency gain from the tax is at least $(C-2(1-\delta))(K-(J-1))$.

5.2 Enforcing Efficiency Using a Decentralized Mechanism

Of course, in many situations, this type of central oversight may be impractical or impossible, so it is also interesting to consider whether it is possible to enforce efficiency through a decentralized mechanism.

Definition. *For any finite game of perfect information we can define a variant game with contingent payments. The game with contingent payments is identical to the original game, but before every decision by Player i in the original game, all other players, in order, have the opportunity to offer Player i utility transfers contingent on his next action. We assume that the order in which players have the opportunity to offer transfers is common knowledge.*

Proposition 7. *The outcome which maximizes total welfare, in any finite game of perfect information with a unique efficient outcome, is the unique result of a SPE of the variant game with contingent payment.*

Proof of Proposition 7: We begin by proving that the result holds for a simple game with only one action and then extend to general, extensive form games.

Lemma 4. *Consider a game with J players indexed $j\in\{0,1,2,3,...,J\}$. After all other players have the opportunity to offer him contingent transfers, Player 0 will take an action which generates a payoff for all players. Player 0's actions are labeled a and come from the set A. During the bidding process, each of Players 1 through J in order offer Player 0 a set of contingent positive transfers, one transfer for every move available to Player 0. Player 0 will always choose a move which maximizes the total sum of the payoffs for all players.*

Proof of Lemma 4: See Appendix A.3.

We use Lemma 4 to prove Proposition 7. The logic follows the same pattern as the proof of the existence of a SPE in finite games of perfect information. Consider some extensive form game which has been modified by adding sequential contingent transfer offers before each action node, as in the static case.

Assume that at every terminal decision node (nodes which cannot be followed by additional decisions), the current player plays the SPE described in the static case. Every terminal node will then yield a final payout which provides maximum total welfare over all payouts accessible from that decision node.

We can then replace the game with a game which has the same SPE payoff by replacing all terminal nodes with the highest total welfare payoffs accessible from those terminal nodes. Then repeat the process with the terminal nodes of the new game. This process can be repeated until no terminal nodes remain to find the outcome of a SPE of the game. Note that only an outcome which provides the highest total welfare will survive this process, since such an outcome can only be eliminated by another welfare maximizing outcome. □

The usefulness of the proposition depends on whether it is possible to implement the variant game with contingent payments, which in turn critically depends on the possibility of transfers.

6 Conclusion

In this paper we explore how including strategic dynamics can enrich network formation. We introduce and analyze a model of network growth incorporating history dependence and strategic agents. This combination of features allows us to explore interesting dynamic questions such as how entry timing can impact node centrality and why many economic systems are dominated by early, but not first, entrants.

In our model, we find that only vying for dominance or pseudo-myopic play will ever be optimal. Even when players break ties in favor of connecting to newer nodes, the period between opportunities for vying increases exponentially as the game progresses, meaning that earlier players will dominate the network.

The decrease in the frequency of strategic vying behavior comes from the increasing cost of vying over time. As the network grows and the most central nodes become more central, competing for centrality becomes more costly. The benefits of vying therefore must also grow in order for vying to be profitable for later players. Since vying players compete with each other for beneficial future connections, in order for benefits to increase, in equilibrium the number of vying players must decrease over time.

We also consider a model where connection behavior is less restricted. For this model we show that certain degenerate networks form when the cost of connections is particularly high or low. While this general pattern is similar to results in static network formation games, the details are distinct. In particular, the high potential future benefits of vying behavior in our model enable the formation of networks with more than the minimum number of connections even at very high connection cost levels. Even in the less restricted game, we find that opportunities for profitable multiple connection play quickly die out as network formation progresses. This is the cause of the early mover advantage discussed previously.

We also consider the efficiency of the networks that form. While the efficiency of networks is not impacted by the dynamic nature of our model, the set of possible outcome networks is affected. Notably, the inclusion of dynamics allows for inefficient over-connection in addition to the inefficient under-connection seen in other models. Our final results provide both centralized and decentralized mechanisms for enforcing efficient outcomes in finite games of perfect information such as the network formation games discussed in this paper.

The phenomenon of vying and the resulting early mover advantage likely apply to more general systems than those discussed in the main text. In Appendix B we explore several extensions to the base game in which certain key assumptions of the model are relaxed. These extensions examine how results change when players are not required to form connections, connections have heterogeneous costs, players may own multiple nodes, or the move order is more flexible than in the main game. In general, the phenomenon of vying for dominance can still arise as long as connections are relatively static, centrality is beneficial, and the network is growing over time. As such, behavior that is analogous to vying for dominance may arise in many real-world networks of interest.

In transportation networks, connections are generally static and centrality can garner more traffic and profit. If a developer or city wanted a new train hub to be highly central in the future, they may want to over-invest in current connections by adding more lines or routes than would initially be required. This investment would encourage later routes and lines to go through the hub.

In professional networks, developing relationships with colleagues, managers, vendors, and specialists requires a significant upfront investment. As in our model, these relationship costs may fall disproportionately on new entrants to the network, such as start-up businesses or recent graduates. Therefore, it might be beneficial to accrue a large number of connections early in one's career in order to become an attractive target for new individuals to approach in the future.

Currently, the less restricted variant of our game is not tractable for existing real-world networks. In addition, continual games with nodes being constantly added to the network may be more realistic for some applications. In such situations folk theorem variants would be needed to explore the set of solutions. Future work must also investigate how the SPEs of the base game function for large networks at intermediate cost levels. There are complex strategic interactions that can occur in response to vying for dominance, as seen in Appendix C.1, which are not well captured by the baseline model.

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References

Akerlof, R. and Holden, R. (2016). Movers and shakers. *Quarterly Journal of Economics*, 131(4):1849–1874.

Albert-László, Albert, R., and Hawoong, J. (2000). Scale-free characteristics of random networks: the topology of the world-wide web. *Physica A: Statistical Mechanics and its Applications*, 281(1-4):69–77.

Allouch, N. (2015). On the private provision of public goods in networks. *Journal of Economic Theory*, 157:527–552.

Apt, K. R., de Keijzer, B., Rahn, M., Schaefer, G., and Simon, S. (2016). Coordination games on graphs. *International Journal of Game Theory*. Forthcoming.

Aumann, R. and Myerson, R. (1988). Endogenous formation of links between players and coalitions: An application of the shapley value. In Roth, A., editor, *The Shapley Value*. Cambridge University Press, Cambridge.

Bala, V. and Goyal, S. (2000). A noncooperative model of network formation. *Econometrica*, 68(5):1181–1229.

Blume, L., Easly, D., Kleinberg, J., and Tardos, E. (2009). Trading networks with price-setting agents. *Games and Economic Behavior*, 67(3):36–50.

Bourlès, R., Bramoullé, Y., and Perez-Richet, E. (2017). Altruism in networks. *Econometrica*, 85(2):675–689.

Bramoullé, Y. and Kranton, R. (2007). Public goods in networks. *Journal of Economic Theory*, 135(1):478–494.

Calvó-Armengol, A., de Martí, J., and Prat, A. (2015). Communications and influence. *Theoretical Economics*, 10(2):649–690.

Candogan, O., Bimpikis, K., and Ozdaglar, A. (2012). Optimal pricing in networks with externalities. *Operations Research*, 60(4):883–905.

Carpenter, J., Kariv, S., and Schotter, A. (2012). Network architecture, cooperation and punishment in public good experiments. *Games and Economic Behavior*, 73(1):76–90.

Carrillo, J. D. and Gaduh, A. (2012). The strategic formation of networks: Experimental evidence. *CEPR discussion paper*.

Chen, W. and Teng, S. H. (2016). Interplay between social influence and network centrality: A comparative study on shapley centrality and single-node-influence centrality. *CoRR*, abs/1602.03780.

Chowdhury, A. (2008). Essays in network formation (doctoral dissertation).

Claussen, J., Kretschmer, T., and Spengler, T. (2010). Backward compatibility to sustain market dominance - evidence from the us handheld video game industry. Discussion paper, Munich School of Management.

Corominas-Bosch, M. (2004). Bargaining in a network of buyers and sellers. *Journal of Economic Theory*, 115(1):35–77.

Currarini, S. and Morelli, M. (2000). Network formation with sequential demands. *Royal Holloway, University of London: Discussion Papers in Economics*, (99/2).

Duernecker, G. and Vega-Redondo, F. (2017). Social networks and the process of globalization. *Review of Economic Studies*, 85(31):1716–1751.

Dutta, B., Ghosala, S., and Ray, D. (2005). Farsighted network formation. *Journal of Economic Theory*, 122:143–164.

Eiron, N. and McCurley, K. S. (2003). Locality, hierarchy, and bidirectionality in the web. In *presented at the Workshop on Algorithms and Models for the Web Graph, WAW*, Budpest, Hungary.

Goyal, S. (2007). *Connections: an Introduction to the Economics of Networks*. Princeton University Press.

Herings, J.-J., Mauleon, A., and Vannetelbosch, V. (2009). Farsightedly stable networks. *Games and Economic Behavior*, 67(2):526–541.

House, T., Reed, J., Danon, L., and Keeling, M. (2015). Testing the hypothesis of preferential attachment in social network formation. *EPJ Data Science*, 4(13).

Jackson, M. (2003). A survey of models of network formation: Stability and efficiency. *Game Theory and Information*, (0303011).

Jackson, M. (2008). *Social and Economic Networks*. Princeton UNiversity Press.

Jackson, M. and Wolinsky, A. (1996). A strategic model of social and economic networks. *Journal of Economic Theory*, 71:44–74.

Kim, J.-Y. and Jo, H.-H. (2009). A signaling explanation for preferential attachment in the evolution of social networks. *Journal of Evolutionary Economics*, 20:375–393.

Kondor, D., Pósfai, M., Csabai, I., and Vattay, G. (2014). Correction. sdo the rich get richer? an empirical analysis of the bitcoin transaction network. *PLoS ONE*, 9(5). e97205.

Kranton, R. and Minehart, D. (2001). A theory of buyer-seller networks. *American Economic Review*, 91:485–508.

Maskin, E. and Tirole, J. (2001). Markov perfect equilibrium. *Journal of Economic Theory*, 100:191–219.

McCubbins, M. and Weller, N. (2012). Effects of network structure on costly coordination. *Association for the Advancement of Artificial Intelligence Technical Reports*, (FS-12-08).

Mislove, A., Koppula, H. S., Gummadi, K. P., Druschel, P., and Bhattacharjeet, B. (2013). An empirical validation of growth models for complex networks. In Mukherjee, A., Choudhury, M., Peruani, F., Ganguly, N., and Mitra, B., editors, *Dynamics On and Of Complex Networks, Volume 2*. Modeling and Simulation in Science, Engineering and Technology Series.

Mutuswami, S. and Winter, E. (2002). Subscription mechanisms for network formation. *Journal of Econonmic Theory*, 106(2).

Neligh, N. (2019). Vying for dominance: An experiment in dynamic network formation. Working Paper.

Newman, M. E. J. (2004). Coauthorship networks and patterns of scientific collaboration. *PNAS*, 101(suppl 1):5200–5205.

Page, F., Wooders, M., and Kamat, S. (2003). Networks and farsight stability. *Warwick Economic Research Papers*.

Pollack, J., Forster, W., Johnson, P., Coy, A., and Molden, D. (2015). Promotion- and prevention-focused networking and its consequences for entrepreneurial success. *Social Psychological and Personality Science*, 6(1):3–12.

Powell, W., Koput, K., and Smith-Doerr, L. (1996). Interorganizational collaboration and the locus of innovation: Networks of learning in biotechnology. *Administratvie Science Quarterly*, 41(1):116–145.

Puffert, D. (2004). Path dependence, network form, and technological change. In Sundstrom, W., Guinnane, T., and Whatley, W., editors, *History Matters: Economic Growth, Technology, and Population*. EH.NET.

Puffert, D. J. (2002). Path dependence in spatial networks: The standardization of railway track gauge. *Explorations in Economic History*, 39(3):282 – 314.

Rossi, A., Blake, D., Timmermann, A., Tonks, I., and Wermers, R. (2015). Network centrality and pension fund performance. *CFR Wokring Papers*. University of Cologne, Centre for Financial Research.

Sarigöl, E., Pfitzner, R., Scholtes, I., Garas, A., and Schweitzer, F. (2014). Predicting scientific success based on coauthorship networks. *EPJ Data Science*, 3(9).

Song, Y. and van der Schaar, M. (2015). Dynamic network formation with foresighted agents. Available online: arXiv:1509.00126v2.

Valente, T., Coronges, K., Lakon, C., and Costenbader, E. (2008). How correlated are network centrality measures? *Connect (Tor)*, 28(1):16–26.

Vazquez, A. (2003). Growing network with local rules: Preferential attachment, clustering hierarchy, degree correlations. *Physical Review*, 67.

Wang, M., Yu, G., and Yu, D. (2008). Measuring the preferential attachment mechanism in citation networks. *Physica A: Statistical Mechanics and its Applications*, 387(18):4692–4698.

Watts, A. (2001). A dynamic model of network formation. *Games and Economic Behavior*, 34:331–341.

Yule, G. U. (1925). A mathematical theory of evolution, based on the conclusions of dr. j. c. willis, f.r.s. *Philosophical Transactions of the Royal Society of London B: Biological Sciences*, 213(402-410):21–87.

A Omitted Proofs

A.1 Proof of Proposition 6

To begin, note that all networks have at least $J-1$ connections, because all players after the first must make at least one connect. Next, note that the most efficient network with any fixed number of connections $N \geq J-1$ must contain a star (if the network has fewer than $J-1$ connections it cannot contain a star). A network with N connections containing a star has $2N$ minimum paths of length one, and all other minimum paths are of length two. It is impossible to have more minimum paths of length one than $2N$ minimum paths of length one, so no configuration can produce higher centrality benefits. All networks with N connections produce a total connection cost of $C * N$

Given that the efficient network must contain a star, every additional increases the total centrality benefit of the network by $2(1-\delta)$, because each connection moves two nodes from a distance of two to a distance of one. The social cost of connections is C, so when $2(1-\delta) < C$, the feasible network containing a star with the minimum number of connections is socially optimal. The two networks that fit that criterion are the star network centered on Node 1 and the star network centered on Node 2. Similarly, when $2(1-\delta) > C$, the feasible network containing a star with the maximum number of connections is socially optimal. The network which fits this criterion is the complete network. $\square$

A.2 Proof of Proposition 4 Part 2

Assume $C \geq 1$, because if the alternative is true and δ is sufficiently small, then the result falls under Proposition 3 and we are done. We begin with the following lemma

Lemma 5. *There exists a Player j, who satisfies the following conditions:*
(a) $(J-1)-(j-1)C \geq 1-C$
and
(b) $(J-2)-(j-1)C \leq 1-C$

Proof of Lemma 5: If $C < J-2$ by the assumption that we know $\exists j \in [3, J)$ satisfying condition (a) of Lemma 1. To see this note that If $C < J-2$ guarantees the condition for $j=3$.

Call the last j satisfying condition (a) player j^*. Note that we must have $j^* < J$, because condition (a) reduces to $(J-1) \leq 1$ when $j = J$ as long as $C > 1$, which cannot hold when $J > 3$

We know $\frac{J-2}{j} > C$ from the fact that j satisfies condition (a) and $\frac{J-2}{j-1} < C$ from the fact that Player j^*+1 does not.

We now prove that condition (b) of Lemma 5 also holds for j^* by contradiction. Assume that condition (b) does not hold for j, so $\frac{J-3}{j} > C$. The two conditions $\frac{J-3}{j} > C$ and $\frac{J-2}{j-1} < C$ combine to give us $\frac{j-1}{j} > \frac{J-2}{J-3}$ which contradicts $j < J$. As such, (b) must hold for j^*. $\square$

Consider the move of Player j^* from Lemma 5 in the equilibrium where players resolve indifferences in favor of connecting to newer nodes. If Player j^* connects to all existing players, he will receive connections from all future nodes. We show this by backwards induction using the following induction hypothesis

Induction Hypothesis 4: All players after Player $J-k > j^*$ will connect to the newest dominant node.

Assume that j^*connected to all existing nodes as he joined the network. We need to prove that under Induction Hypothesis 4, Player $J-k$ will also connect to the newest dominant node. Note that J satisfies Induction Hypothesis 4, so that requirement for backwards induction is fulfilled.

First we show that under Induction Hypothesis 4, Player $J-k$ will either connect to the newest dominant node or become the newest dominant node. Later we will show that becoming the newest dominant node is never optimal.

All moves other than connecting to a dominant node or vying for dominance are strictly worse as long as δ is small. The benefit of any move which does not generate future free connections converges to $N(1-C)$ as δ goes to zero where N is the number of connections made. Given $C > 1$, $N(1-C)$

is decreasing in N. This means that if δ is sufficiently small, the benefit of the optimal N connection move which does not lead to future connections must be decreasing in N.

The optimal one connection move is always connecting to a dominant node, because doing so provides the maximum immediate benefit (by definition) and also the maximum benefit from future connections. All future players will be a distance two away, because they will connect to the same dominant node.

Only moves which result in Player $J-k$ becoming a dominant node will provide additional connections under Induction Hypothesis 4.

We will show that vying for dominance is never optimal for player $J-k > j^*$. The minimum number of connections required to become a dominant node for Player $J-k$ is j^*-1, since j^* has a centrality of at least j^*-1 and making fewer than j^*-1 connections will produce a centrality less than j^*-1 when δ is small.

This means that the best possible net gain that a Player $J-k > j^*$ can get from becoming a dominant node is gained by Player $J-k = j^*+1$ connecting to all nodes that Player j^* connected to. By doing this, Player j^*+1 gets $(J-2)-(j-1)C+\delta$. By making a single connection to Player j^*, a Player $J-k > j^*$ can get $1-C+g(\delta, G_{J-k})$ where $g(\delta, G_{J-k})$ is some function representing gains from indirect connections to other nodes. Notably $g(\delta, G_{J-k}) \geq \delta$. As such, thanks to condition (2), Player $J-k > j^*$ will always connect to a single dominant node rather than becoming a dominant node

We now return to considering the move of Player j^*. It is better for Player j^* to connect to all nodes and gain direct connection from all future players rather than connecting to a single dominant node. To see this, note that connecting to all nodes is preferred as long as $(J-1)-(j-1)C > 1+(J-2)\delta-C$. This condition always holds if Lemma 5 condition (a) holds and δ is sufficiently small. Therefore if δ is sufficiently small, then the SPE of the game in which players break ties in favor of connecting to the newest node cannot form a star network with certainty. Otherwise Player j^* would have a profitable deviation. $\square$

A.3 Proof of Lemma 4

We prove this by induction.

Player J:

Begin by considering the behavior of Player J. During his move the contingent transfers plus baseline payoffs for Player 0 sum to provide a set of payoffs which we will refer to as π_0^J, in other words the payoffs for Player 0 before Player J bids. Player J also has a set of payoffs π_J. It is trivial to see that it is a weakly dominated move to offer transfers contingent on multiple moves, so we can assume Player J is offering a transfer for one move equal to t. Player J's problem is then

$$\max_{t,a} \pi_J(a) - t$$

$$s.t. \pi_0^J(a) + t \geq \pi_0^J(a_J^*)$$

where

$$a_J^* = \arg\max_{a \in A} \pi_0^J(a)$$

Since Player J will always choose to implement the desired actions with the minimum possible transfer, the optimization problem can be rewritten

$$\max_a \pi_J(a) + \pi_0^J(a) - \pi_0^J(a_J^*)$$

Then it is optimal for Player J to implement

$$\hat{a} = \arg\max_{a \in A} \pi_0^J(a) + \pi_J(A)$$

with a transfer of $\pi_0^J(a^*) - \pi_0^J(\hat{a})$. Note that this means if

$$\arg\max_{a\in A} \pi_0^J(a) = \arg\max_{a\in A} \pi_0^J(a) + \pi_J(A)$$

Then Player J cannot profitably change Player 0's action, since the cost of the transfer would be greater than the possible points gained. However, if

$$\arg\max_{a\in A} \pi_0^J(a) \neq \arg\max_{a\in A} \pi_0^J(a) + \pi_J(A)$$

then it is profitable to change Player 0's action

Player J-k: We now discuss the behavior of earlier players using the following induction hypothesis

Induction Hypothesis 5: All later players $J-k+j$ will choose to offer a transfer of

$$\pi_0^{J-k+j}(a^*_{J-k+j}) + \sum_{i=J-k+j+1}^{J} \pi_i(a^*_{J-k+j}) - \pi_0^{J-k+j}(\hat{a}_{J-k+j}) + \sum_{i=J-k+j+1}^{J} \pi_i(\hat{a}_{J-k+j})$$

For action $\hat{a}_{J-k+j}$, where

$$a^*_{J-k+j} = \arg\max_{a\in A} \pi_0^{J-k+j}(a) + \sum_{i=J-k+j+1}^{J} \pi_i(a)$$

and

$$\hat{a}_{J-k+j} = \arg\max_{a\in A} \pi_0^{J-k+j}(a) + \sum_{i=J-k+j}^{J} \pi_i(a)$$

Under Induction Hypothesis 5, Player $J-k$'s decision problem can be written as

$$\max_{t,a} \pi_{J-k}(a) - t$$

$$s.t. \pi_0^{J-k}(a) + \sum_{i=J-k+1}^{J} \pi_i(a) + t \geq \pi_0^{J-k}(a^*_{J-k}) + \sum_{i=J-k+1}^{J} \pi_i(a^*_{J-k})$$

where

$$a^*_{J-k} = \arg\max_{a\in A} \pi_0^{J-k}(a) + \sum_{i=J-k+1}^{J} \pi_i(a)$$

Which means that, similar to Player J, Player $J-k$ will choose to offer a transfer of

$$\pi_0^{J-k}(a^*_{J-k}) + \sum_{i=J-k+1}^{J} \pi_i(a^*_{J-k}) - \pi_0^{J-k}(\hat{a}_{J-k}) + \sum_{i=J-k+1}^{J} \pi_i(\hat{a}_{J-k})$$

For action $\hat{a}_{J-k}$, where

$$\hat{a}_{J-k} = \arg\max_{a\in A} \pi_0^{J-k}(a) + \sum_{i=J-k}^{J} \pi_i(a)$$

Thus by induction, every player j will choose offer a

$$\pi_0^j(a^*_j) + \sum_{i=j+1}^{J} \pi_i(a^*_j) - \pi_0^j(\hat{a}_j) + \sum_{i=j}^{J} \pi_i(\hat{a}_j)$$

For action $\hat{a}_j$, resulting in that action's implementation. To finish the proof for this lemma, consider what this means for Player 1.

Player 1 will offer a transfer which guarantees the implementation of

$$\hat{a}_1 = \arg\max_{a\in A} \pi_0^1(a) + \sum_{i=1}^{J} \pi_i(a) = \arg\max_{a\in A} \sum_{i=0}^{J} \pi_i(a)$$

□

B Further Extensions and Generalizations

In this section we cover a number of extensions to the LRG which make it even more general. The major qualitative results are fairly robust to these generalizations, but the details of solutions to the game can often change a great deal.

B.1 No Connection Requirement

Many readers are likely curious about what happens when we do not require players to make and connections. The answer looks fairly similar to what we see with the base game, although there are some added wrinkles. Define the A Variant Game as the base game but without the restriction that $\boldsymbol{h}_j$ must be non-empty.

Proposition A 6: *Efficient networks in the A Variant Game. are characterized as follows*

1. *If $C < 2(1-\delta)$, then the complete network is the unique efficient network.*
2. *If $2 + (J-2)\delta > C > 2(1-\delta)$, then the star networks (on Node 1 and Node 2) are the only efficient networks*
3. *If $C > 2 + (J-2)\delta$, then the empty network is the efficient network.*

Proof of Proposition A 6: This Proposition and its proof are identical to Proposition 5.5 of Bala and Goyal (2000)

Proposition A 3: *If $C < (1-\delta)$ then the complete network is the only network which can be formed in SPEs of the A Variant Game.*

Proof of Proposition A 3: Identical to the proof of Proposition 2.

Proposition A 4: *If $C > 1 - C + (J-2)\delta$, then the empty network is the only network which can be formed in SPEs of the A Variant Game.*

Proof of A 4: Consider Player J. The most points that Player J can from a single connection is $1-C+(J-2)\delta$ which he can receive by making one connection to a node connected directly to all other nodes. When C exceeds this amount, Player J will never make a connection regardless of network shape. Knowing that Player J will not connect, Player $J-1$ can make no more than $B-C+(J-3)\delta B$ points from making a move with a non-empty set of connections. Player $J-1$ will then not make any connections.

Knowing that all future players will make no connections, Player $J-k$ will make at most $1 - C + (J-k-2)\delta$ points by making a connection. Therefore by induction, no player will make any connections. □

We also get an entirely new result related to connected components of the solution network. First we need a definition.

Definition: *A connected component of a graph is a subgraph such that every node in the subgraph is connected by some path to every other node in the subgraph, and any node which is connected to a node in the subgraph is in the subgraph.*

Proposition A 4: *The second largest connected component of a network formed by a SPE of the A Variant Game must contain fewer than k nodes where k is the lowest integer such that $1 + \sum_{i=1}^{k-1} \delta^{roundup(i/2)} > C$.*

Proof of Proposition A 4: Consider Player J, who only cares about the myopic benefits of his move. When Player J makes his first connections to a node in a specific connected component, the minimum benefit he can gain comes when the subset of nodes are connected in a chain. In that case Player J makes at least $1 + \sum_{i=1}^{k-1} \delta^{roundup(i/2)}$ from connecting. If this value is greater than C, then such a connection will be made.

Note that the benefit that Player J receives from connecting to the first node in a connected component is independent of any other connected components Player J may be connected to. As such, any connected component with k nodes or more will be connected to node J which means it is impossible for the game to conclude with more than one such connected component. If more such connected components existed, Player J would have linked them together. $\square$

Corollary A 1: *When $C < 1$ only connected networks can be formed with positive probability as outcomes of SPEs of the A Variant Game.*

Proof of Corollary A 1: When $C < 1$, by Proposition A3, the second largest connected component must contain fewer than one nodes. Therefore, the second largest connected component must be empty, so the graph must be connected. $\square$

It is natural to wonder what removing the connection requirement does in relation to vying for dominance in this game. In general, the game is much more complex with many more possible outcomes. One type of behavior that can arise in the A Variant Game which cannot arise in the base game arises when $C \in (1-\delta, 1)$. In this case when δ is high, it is often beneficial for the first three nodes to make no connections, leaving Player 4 to connect to all other nodes on the network.

Vying for dominance can still play a major roll, however.

Consider a four node example where ties are broken in a uniform random way. Assume $C = 1 + \epsilon$ where $\epsilon < \delta$ with both ϵ and δ small. Player 4 will not connect to disconnected node, because $C > 1$.

If nodes are connected, Player 4 will connect to the most central, because doing so provides a benefit of $1 + \delta$ or $1 + 2\delta$ depending on how many nodes are connected. In either case, additional connections are not beneficial, since $C > 1 - \delta$.

If ϵ and δ are sufficiently small, Player 3 will always vie for dominance by connecting to all existing nodes. There are two cases we must consider. Either Player 3 faces two disconnected nodes or two connected nodes. If nodes are disconnected, Player 3 can connect to zero (earning 0), one node (earning $1.5 + 0.5\delta - C$ in expectation), or both (earning $3 - 2C$). When ϵ is small, the latter two payoffs are positive. Connecting to both is preferred if $1.5 - 0.5\delta - C > 0$ or $\delta + 2\epsilon < 1$. Since $\delta < 1$, this means Player 3 will connect to both nodes as long as ϵ is small

If existing nodes are connected, Player 3 can connect to zero (earning 0), one (earning $1 + 2\delta - C$), or both (earning $2\frac{1}{3} + \frac{2}{3}\delta - 2C$). Again, the latter two options are preferred when ϵ is small. Player 3 will connect prefer to make two connections if $1\frac{1}{3} - 1\frac{1}{3}\delta - C > 0$ or $4\delta + 3\epsilon < 1$. Therefore, Player 3 will connect to both nodes when ϵ and δ are small.

Given that Player 3 will vie by connecting to both nodes in either case, Player 2 will connect to Player 1 as long as $2\frac{1}{3} + \frac{2}{3}\delta - C > 1 + 2\delta$ or $4\delta + 3\epsilon < 1$.

Therefore, in this example, Player 3 vies, and the outcome looks very similar to the four node vying example in the base game.

B.2 Heterogeneity and Joint Node Ownership

In many interesting systems, nodes will have heterogeneous costs and benefits from connections. Consider a network formed from video games and gaming platforms. Games derive popularity from being available on popular platforms and platforms derive popularity from having popular games. Effectively both games and platforms derive benefits from centrality. However, in such a setup it is generally not beneficial or not possible for games to connect to games or for platforms to connect to other platforms. Hence there is heterogeneity in costs and benefits of connections.

In general, heterogeneity in the benefits and costs of connection does not have a substantial impact on the results. Replace payoffs in the base game with

$$u_i = Y - C_{ij} \cdot |\mathbf{h}_i| + \sum_{j \neq i} B_{ij} \delta^{d_{ij}(G_J)-1}$$

Where B_{ij} is a pairwise benefit parameter. Call the resulting game the B Variant Game. Then we still have analogs of Propositions 1, 2, and 3 as long as the B_{ij}'s all satisfy the corresponding restrictions.

Proposition B 6: *Say $i > j$. Efficient networks in the B variant game are characterized as follows*

1. *If $C_{ij} < (1-\delta)B_{ij} + (1-\delta)B_{ji} \forall i, j$, then the complete network is the unique efficient network.*
2. *If $C_{ij} > (1-\delta)B_{ij} + (1-\delta)B_{ji} \forall i, j$, then the star network (on Node 1 or Node 2) are the only the efficient networks.*

Proposition B 3: *The following rules characterize the formation of complete networks in the B variant game*

1. *If $C_{ij} < (1-\delta)B_{ij} \forall i, j$ then the complete network is the unique network which can form in SPE.*
2. *If $C_{ij} > (1-\delta)B_{ij} \forall i, j$, then the complete network cannot be formed by any SPE.*

Proposition B 4: *If $C_{ij} > \left((J-1) - \frac{1-\delta^{J-3}}{1-\delta}\right) B_{ij} \forall i, j$, then the star networks centered on Node 1 and Node 2 are the only networks which can form any in SPEs of the B variant game.*

In all cases proofs are omitted, because they are effectively identical to the Proofs of Propositions 1, 2, and 3 in the base game. Since the base game is a special case of the B Variant Game, naturally vying for dominance can arise in this variant.

In this type of environment, vying analogous behavior can naturally arise. For example, Claussen et al. (2010) discuss how a new platform could ship with backwards compatibility for many older games in order to get popularity and potentially secure more newer titles.

Another natural generalization for the base game involves allowing a single agent to control multiple nodes. Although we will not discuss this modification in depth, one of the primary effects of this modification is that it introduces heterogeneity of benefits. When a player makes a connection to one of his earlier nodes, he gets twice the benefit. Adding multiple node ownership can also influence a Player's decision to vie for dominance if his own node is currently dominant.

B.3 Multiple Moves

There is another question which cannot be addressed quite so easily: what happens if we relax the restriction that each player only moves as he joins the network? Let's say that each Player i who has already joined the network has a p_{ij} chance of getting the opportunity to add connections again after each new Player j joins the network. Moving again means that a player has the opportunity to make a connection to any node in the network that he is not currently connected to. The resulting possible moves occur in some known order. The random chance drawn and revealed immediately after Player j's move. Call this version of the game the C Variant Game.

Proposition 1 from the base game still applies to the C Variant Game, because payoffs have not changed, and the set of feasible networks has not changed.

Proposition 3 also holds by the same logic as before. No player wants to make any extra connections when costs are so high that he could not possibly benefit from doing so.

Proposition 2 is slightly more complicated. If the move order is deterministic (with $p_{ij} \in 0, 1$), each player would always want to make every connection that no later player will be able to make. In this case Proposition 2 would hold as in the base game.

However, if some Player i knows that another Player j might have an opportunity to make connection (i, j) later in the game, Player i may hold off, knowing that Player j will make the connection if he can.

Consider what happens when $C = 1 - \delta - \epsilon$ and there is some Player i with $1 > p_{iJ} > 0$. Assume that all other p_{jk}'s are zero. In this case if ϵ is sufficiently small and p_{iJ} is sufficiently high, Player J will prefer to not connect to Player i. If Player J makes the connection, he receives a benefit of $1 - C = \epsilon$ from that connection. If he does not connect, Player i may get to move, and in that case Player i will choose to connect to Player J. Therefore, Player J's expected benefit from not moving is $p_{iJ} - \epsilon$, which can easily be greater than zero.

If Player J chooses not to connect to Node i and gets unlucky in this case, then the complete network will not form. The complete network will always have a positive chance of forming, however, since any player who knows for certain that he is the last player who can make a specific connection will always do so.

Example: While vying for dominance remains important in the C Variant Game, the nature of vying can change. Again consider the four node example. This time assume that every player has the opportunity to purchase connections after every new player joins as long as they are part of the network. For the sake of simplicity assume the additional connection opportunities go in network joining order

Assume $C \in (1 - \delta^2, 1\frac{1}{3}(1 - \delta))$. $C > 1 - \delta^2$ guarantees that the Player 3 will not make any connections after Player 4 joins the network, since the benefits from doing so will not be worth the cost. Knowing this Player 2 will not make any connections after Player 4 joins and neither will Player 1.

No Players will make connections after Player 4, and Player 4 will therefore behave myopically as in the base game.

It is therefore only important to consider Player 3's move and the connections made after it. Player 3 can make one connection or no connections. If Player 3 is not connected to Player 2 when Player 2 has his opportunity to make extra connections, Player 2 will connect, because $C < 1\frac{1}{3}(1 - \delta)$. Making that connection will always make Player 2 a dominant node, so the connection will always provide a benefit of at least $1\frac{1}{3}(1 - \delta)$.

For similar reasons, Player 1 will connect to Player 3 if he has the opportunity to do so after Player 3 joins. Notably, this means that Player 3 has no incentive to make multiple connections. He will connect to either Node 1 or Node 2 and receive the other connection for free.

In equilibrium we will therefore see Player 2 connect to Node 1. Then Player 3 connects to either Node 1 or Node 2 at random. Then the other Player (either 1 or 2) connects to Node 3. Finally Player 4 connects to one node at random. The result looks quite similar to the base game example, but the nature of vying has changed. Now it is either Player 2 or Player 1 who is vying by making more connections than he myopically should in order to become dominant and secure a potential connection from Player 4. Player 3 essentially gets to free-ride on this vying.

C Limitations and Counter Examples

C.1 Complex Strategic Behaviors Example

In this section, we provide an example that demonstrates two counter-intuitive properties which can make it difficult to make definitive statements about outcomes in the non-degenerate region in the baseline game. The first property is that nodes do not always move myopically or vie for dominance. While in the small example discussed in Section 2.3, these are always the best options, in larger networks, more complex behaviors can arise. This can make characterizing equilibria exceptionally difficult

The second property is that it is not always possible to support a star network by using a tie-breaking rule which favors older nodes. In the Section 2.3 example, if players break ties in favor of earlier nodes, only the star network can form, but this is not always the case with larger networks. As such, it is difficult to determine when the star network or is not a possible result for a SPE of the game.

Consider what happens in the case where $J = 6, B = 1, \delta = 0.05, C = 1.1$, and all players break ties in favor of connecting to the set of nodes with the greatest total age.

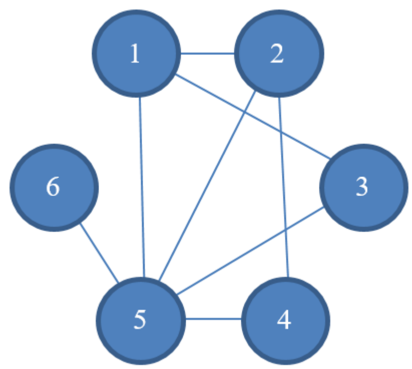

Figure 8: Graph generated by stability seeking tie-breaking rule when $B = 1, \delta = 0.05, C = 1.1$.

Player 6 will connect to the oldest dominant node, since the gain from a second connection can never compensate for the cost, and he breaks ties in favor of older nodes.

Player 5 will connect to all existing nodes if doing so will make him the only dominant node. Otherwise, he will connect to a single dominant node as well, because if he is not the only dominant node, he will not receive a connection from Player 6. If Player 5 cannot receive a future connection, a myopic move is his best choice. Player 5 can only become the sole dominant node if he is facing a chain or a box.

Player 4 knows this, so he will create the chain if available, since doing so requires only one connection and gets him a future connection from Player 5. Creating the box also provides a future connection but requires Player 4 to purchase one more connection, which is not worth the cost. No other moves provide a future connection for him. By the tie-breaking method, Player 4 will always connect to the oldest node of any three node chain he faces to create the longer chain (which will always be Node 1 or Node 2).

If he cannot form a chain (since he is facing a complete network), Player 4 will connect to the oldest dominant node, which can never be Node 3.

Player 3, then will just connect to one node, since doing so will have minimal cost and provide one future connection from Player 5. Connecting to two nodes will provide no future connections. As such, in the resulting equilibrium Player 3 will connect to Node 1, Player 4 will connect to Node 2, Player 5 will connect to all existing nodes, and Player 6 will connect to Node 5 (see 8).

Note that Player 4 in this case is not choosing a myopic move or vying for dominance. In addition, the star network does not form even though players are using a tie-breaking rule which favors older nodes. In Part 3 we examine a slightly modified version of the game in which these phenomena do not arise.

Given the strategic complexity of the game, it is natural to consider whether the game can be easily solved using simulation methods. Unfortunately, this is impractical for networks of any significant size

While we can solve the game for small networks (up to around 6 nodes in size), solving larger games is very difficult. Brute force backwards induction in a J node network would require looking at the payoffs associated with $\prod_{j=2}^{J} 2^{j-1} - 1 > \prod_{j=2}^{J} 2^{j-2} = 2^{(j-1)(j-2)/2}$ possible networks. Each individual can choose whether or not to connect to any existing node (2^{j-1} choices), but he cannot choose to connect to no nodes (hence the -1). It may be possible that some simplifications may be made which reduce the computational complexity, but in general, the range and sophistication of strategic behaviors grows rapidly in complexity with the size of the network.

We can take a number of approaches to exploring the game in the unknown region, all of which will be explored at various points in the paper. In Part 3 we consider a more tractable modified version of the game. In the restricted game, players are required to connect to at least one dominant node as part of their moves. Surprisingly, this restriction is enough to get us tractability and allow us to pin down precisely the critical times at which nodes should vie and become dominant.

C.2 Importance and Limitations of Markov Perfection

Markov perfection is important, because it prevents certain types of unusual sunspot equilibria where players condition the way that they resolve indifferences on payoff irrelevant features of the game history.

Consider an example where Player 6 will never connect to Player 5 unless Player 4 connects only to Player 1 and Player 2 (an alternative move that is neither a pseudo-myopic move nor vying for dominance). If Player 4 makes that specific move, Player 6 will always connect to node 5 when doing so is optimal. Player 6 is indifferent, so he can freely condition his moves this way. Assume all other Players will always resolve indifferences in favor of connecting to Player 1. Set $C = 1 - \delta + \epsilon$.

Most moves Player 4 can choose will lead to no future connections, because he cannot get future connections by vying for dominance given the way indifferences are resolved. However, if Player 4 makes the specified move, he can change the incentives of Player 5 such that is is now in Player 5's interest to vie for dominance. This means that Player 4 will get an extra future connection from this alternative move relative to all other moves, which makes it worth the extra cost relative to just connecting to one dominant node. This unusual move does not influence the payoffs of any move for Players 5 and 6, but due to the way Player 6 is conditioning his tie-breaking, it does change the network formation process substantially. A move that is payoff irrelevant can change the behavior of future players by changing the way that other players even further down the line resolve indifferences.

It may seem unusual that pseudo-myopic moves involve connecting to a subset of dominant nodes rather than a single dominant node. The reason for this is two-fold. First, Markov Perfection does not prevent sunspots which depend on the set of dominant nodes. To see this, note that under the definition of Maskin and Tirole (2001), the partition of histories on which players may condition their moves can be no coarser than the partition which determines the set of available moves to the player. In this game the set of dominant nodes maps one to one with the set of available moves.

Even if we eliminate all sunspots, however, there can still be unusual combination of consistent tie-breaking behaviors which lead to a non-singleton pseudo-myopic moves being optimal. An example is provided in the Appendix C.3. The example is fairly complex, but the basic idea is that sometimes it is a good idea to keep more dominant nodes in the set of dominant nodes to prevent a particularly favored node from vying for dominance, since that node's vying for dominance discourages several nodes from vying for dominance further in the future.

C.3 Pseudo-Myopic Moves

This example shows why, even in MPEs of the baseline game with no sunspots, players sometimes connect to non-singleton subsets of $\mathbf{d}(G_{t-1})$. In this example we have $J = 8, B = 1, C = 1.19$. Player 4 will be optimally choosing to connect to two elements of $\mathbf{d}(G_{t-1})$. Ties are broken in the following manner:

- Player 8 favors Player 6 over Player 7 and Player 7 over everyone else, Otherwise ties are broken at random
- Player 7 breaks ties randomly but never favors node 5
- Player 6 favors Player 5 and otherwise is random
- Player 5 breaks ties randomly
- Player 4 never breaks ties in favor of Player 3
- All other indifferences are resolved randomly, although this is largely unimportant

We solve the game by backwards induction, giving each player's strategy and then a proof of why is optimal given the strategies of later players.

Player 8 will always chooses a myopic move connecting to one node

Proof: Always true of Player J

Player 7 will choose to vie unless Node 6 is dominant. If node 6 is dominant, he will choose a myopic move connecting to a single node. Choice of singleton myopic is payoff irrelevant for Player 7.

Proof: It is profitable for Player 7 to choose to vie when Node 6 is not dominant if $7(1-\delta)-6C > 1-C$, which is true since $\frac{J-2}{J-3} = \frac{6}{5} > \frac{C}{1-\delta} = 1.19$. Vying is always bad when Player 6 is dominant, since Player 7 will never get a connection from Player 8 in that case. We can see that the choice of myopic move is payoff irrelevant, because given any such move, Player 8 will not connect to Player 7.

Player 6 will choose to vie if the number of dominant nodes other than node 5 is less than two. Otherwise he will choose a singleton myopic move. Again choice of myopic move is payoff irrelevant

Proof: If Player 6 becomes dominant then both Player 7 and Player 8 will connect to the same dominant node chosen at random, but never to Node 5 by tie-breaking. Therefor choosing to vie is profitable when $k = 1$ but unprofitable when $k = 2$. Note that if Player 6 picks single connection myopic move, Players 7 and 8 will always pick vying and a myopic move respectively meaning that Player 7's choice of myopic move is not payoff relevant.

Player 5 will vie for dominance if the number of dominant nodes is greater than or equal to 2. Otherwise he will choose a myopic move.

Proof: To see this note that if there are at least two nodes that are dominant, then when Player 5 picks vying, Player 6 will choose a myopic move and connect to Node 5. Player 7 will choose to vie and receive a connection from Player 8.

If Player 5 were to choose a myopic move instead, then Player 6 would choose to vie and then Player 7 and 8 would choose a myopic move. Vying is then preferred if $6(1-\delta) - 4C > 2(1-\delta) - C$ or $\frac{4}{3} > C$, which it is for the given parameters.

If the number of dominant nodes is less than 2 then Player 5 will never get a future connection after choosing to vie, so vying is never optimal, and he will choose a myopic move.

Player 4 will connect to two dominant nodes if there are at least two dominant nodes (a pseudo-myopic move). Otherwise, Player 4 connects to a single dominant node.

Proof: If there are two or more dominant nodes and Player 4 chooses to vie then the resulting moves will take the pattern $VMVM$ (where V represents a vying move and M represents a myopic move). Neither myopic player will connect to Player 4. This is the same pattern the future moves generated if Player 4 connects to only two dominant nodes, so connecting to two nodes is preferred, since it is lower cost. If Player 4 connects to one dominant node the future moves will follow the pattern $MVMM$. Player 4 will then make two connections if $4(1-\delta) - 2C > 2(1-\delta) - C$ or $2(1-\delta) > C$ which it is for the given parameters.

If there are not two dominant nodes then all moves lead to the future move pattern $MVMM$, so connecting to a single dominant node is optimal.

Player 3 Player 3 makes 2 connections

Proof: If Player 3 makes two connections the future move pattern is $MVMVM$, whereas if Player 3 makes one connection, the future move pattern is $MMVMM$. Player 3 will never receive a connection from any of the Move A Players . Since $C < 2(1-\delta)$, it is optimal for Player 3.

So in the end we see the following move pattern $(VV)VPVMV$, where P is the two connection pseudo-myopic move. The first two Players do not actually make choices, but they do end up as dominant nodes, so we consider them to be vying. The special P move happens because Player 4 makes two connections in order to prevent the pattern $(VV)VMMVMM$, which lower his payoff. Essentially, Player 4 picks between the following two move sequences and chooses the latter

Player	1	2	3	4	5	6	7	8
Sequence 1	V	V	V	M	M	V	M	M
Sequence 2	V	V	V	P	V	M	V	M

D Generalized Proposition 2

Definition. *Two newest dominant random (TNDR) tie-breaking requires that when a player is indifferent between multiple pseudo-myopic moves, he will connect to the newest dominant node with a probability of p and the second newest dominant node with a probability of $1-p$ as long as both of those moves are optimal. If only one of those moves is optimal, he will select it with certainly. When connecting to the newest dominant node nor connecting to the second newest dominant node is optimal, the player randomizes uniformly between all optimal moves.*

This tie-breaking rule generates a MPE of the game in which there are periods of pseudo-myopic play separated by instances of vying for dominance, and number of pseudo-myopic moves between vying players follows a predictable pattern. Note that this model includes special cases where the newest dominant node always receives a connection from indifferent pseudo-myopic players and cases where he never receives a connection from indifferent pseudo-myopic players. In the latter case only a star can form, as we shall demonstrate.

Proposition 2*: *Under TNDR, if $1-\delta < C$ the unique solution to the baseline network formation Game is characterized as follows*

1. *All players will either connect to all existing nodes (vie for dominance) or make a single connection to one of the two newest dominant nodes (choose a myopic move).*
2. *If n players vie for dominance during the game, the number of myopic moves between the ith vying player and the $i+1$st is approximately*

$$\lambda^{n-i}\frac{J-1}{p}\left(1-\frac{1-\delta}{C}\right)$$

Where $\lambda = \frac{C(1-p)-(1-2p)(1-\delta)}{C-(1-p)(1-\delta)}$ as long as the ith vying player arrives after period

$$J-\sum_{j=1}^{\tilde{j}}\lambda^{j-1}\frac{J-1}{p}\left(1-\frac{1-\delta}{C}\right)$$

3. *If*

$$1 \geq \max\left(\frac{C+\delta-C\delta p-\delta^2 p-p^2+\delta p^2}{C-(1-p)(1-\delta)}, \frac{1-p\delta}{C-(1-p)(1-\delta)}\right)$$

The difference between the approximation and the true number is less than $n-i+1$.

If we assume TNDR tie-breaking, $1-\delta < C$, and we ignore the integer constraints of node indexing, then in the unique solution to the network formation game, after a certain Node $\bar{j}$ the network formation process is characterized by periods of myopic play separating individual vying players with the time between vying moves growing exponentially.

Proof of Proposition 2*:
Part 1: We begin by characterizing the equilibrium in a general manner.

Lemma 6. *Under the TNDR tie-breaking all players will vie for dominance or pick the special myopic move which involves connecting to the newest dominant node with probability p and connecting to the second newest dominant node with probability $1-p$, and which action they pick does not depend on the current network state.*

Proof of Lemma 6: We show this through backwards induction using the following induction hypothesis.

Induction Hypothesis 6: All future nodes will vie for dominance or choose the special myopic move, and this decision does not depend on the current network.

First we show that Player J satisfies the induction hypothesis, then we show that as long as all later players satisfy the induction hypothesis, so does Player $J-k$.

Player J: As we saw in the proof of Proposition 2, player J will connect to a single $b_{J-1} \in \mathbf{d}(G_{J-1})$. Furthermore, player J will be indifferent between all members of $\mathbf{d}(G_{J-1})$, so by our assumption of TNDR tie-breaking, player J will choose the special myopic move.

Player $J-k$: Now we want to show that, given Induction Hypothesis 6, Player $J-k$ will also always choose the special myopic move or vie for dominance, and this decision does not depend on the current network. We do this by showing that it is weakly optimal for player $J-k$ to choose the special myopic move or strictly optimal for player $J-k$ to vie, and which possibility is true does not depend on the current network. Showing this will guarantee our result, since when the special myopic move is weakly optimal, it will always be selected by TNDR tie-breaking.

Under Induction Hypothesis 6, a player will get a fixed number of future connections for any move that does not put him in $\mathbf{d}(G_{J-k})$ since all future vying players will connect to him and all future myopic movers will not, and these two groups are fixed under the induction hypothesis. Note that if node j is not in $\mathbf{d}(G_j)$ he will never be in $\mathbf{d}(G_t)$. Also note that the only move that will result in node j being part of $\mathbf{d}(G_j)$ is vying for dominance.

By the DNR a player must connect to at least one $b_{t-1} \in \mathbf{d}(G_{t-1})$. Adding any connection to a move will always make the move worse as long as that change does not result in vying for dominance, because doing so increases the connection cost by C and increases centrality benefits by only $1-\delta$ if future moves are unchanged. As such the best move must always be connecting to a single b_{t-1} or vying. Since the set of future connections is fixed for all non-B moves and the special myopic move involves connecting to a single b_{t-1}, the special myopic move is always tied for maximum utility for all non-vying moves.

We have shown that either the special myopic move is weakly optimal or vying is strictly optimal. All that remains is to show that whether vying is strictly better does not depend on the current network. Since we have already shown that the payoff for the special myopic move does not depend on the network, we must now show that vying payoffs are similarly fixed.

Since the connection cost of vying is already fixed and the myopic benefit is as well we must only show that the set of future direct connections to $J-k$ after vying does not depend on the network. By Induction Hypothesis 6, we know that the order of myopic and vying moves is predetermined.

Consider the play after $J-k$ vies. If he is the only dominant node, then he will receive a benefit of $1-\delta$ from every myopic player. If he is one of several dominant nodes, Player $J-k$ has a p chance of receiving a benefit 1 from a myopic player and remaining central if he is the newest dominant. The chance is $1-p$ if he is the second newest dominant node, and 0 otherwise. When he is not a dominant node, Player $J-k$ will receive a benefit of $1-\delta$ from all myopic players. In all cases, Player $J-k$ receives a benefit of 1 from future vying players

If we can show that whether Player $J-k$ is the newest dominant node, second newest, only dominant, or not a dominant node depends only on moves of later players, then we have shown that expected utility from vying does not depend on the current network state and we are done. Note that if Player $J-k$ is dominant but older than the second newest node, he is effectively no longer a dominant node, because he will never again receive connections from myopic players again.

Immediately after vying, Player $J-k$ is the newest dominant node. If Player $J-k$ is the newest dominant node then a new player can either (1.a) play myopically connecting to node $J-k$, (1.b) play myopically connecting to the second newest dominant node, (1.c) vie for dominance. In case (1.a), player $J-k$ becomes the only dominant node. In case (1.b) he is no longer a dominant node and never will be dominant again, and in case (1.c) he becomes the second newest dominant node.

If Player $J-k$ is the only dominant node then a new player can either (2.a) play myopically connecting to node $J-k$ or (2.b) vie for dominance. In case (2.a), Player $J-k$ remains the newest dominant node. In case (2.b) becomes the second newest dominant node.

If Player $J-k$ is the second newest dominant node, then a new player can either (3.a) play myopically connecting to node $J-k$, (3.b) play myopically connecting to the newest dominant node, (3.c) vie for dominance. In case (1.a), player $J-k$ becomes the only dominant node. In case (3.b) he is no longer a dominant node and never will be dominant again, and in case (3.c) he becomes the third newest dominant node, which is effectively the same as outcome 3.b.

Since these transitions only depend on whether later players choose the special myopic move or vie for dominance, and under the induction hypothesis, their choice among these moves does not depend on the current network state, then the benefit from vying does not depend on the current network state.

Therefore the probability of a given sequence of future connections for Player $J-k$ after vying, does not depend on the current network.

By induction all players will choose the special myopic move or vie for dominance and this choice does not depend on the network. □

Part 2: Having characterized the equilibrium in general terms, we now determine which players will choose the special myopic move and which will vie for dominance.

Consider J sufficiently large as to ensure that all the node indices referenced are positive.

As before we work through backwards induction.

Player J: Begin by considering the move of Player J. His expected utility from the special myopic move is $1 + \delta(J-2) - C$, because he will be directly connected to a single node in $\mathbf{d}(G_{J-1})$ and two jumps from all other nodes. As discussed previously, Player J will never vie as long as $C > 1 - \delta$.

Player $J-k$: We now consider Player $J-k$ under the following induction hypothesis

Induction Hypothesis 7: All following players will choose the special myopic move.

By Lemma 6 Player $J-k$ will not influence the choice of future players. If Player $J-k$ vies, Player $J-k+1$ will directly connect to Node $J-k$ with probability p since Node $J-k$ will be the newest node in $\mathbf{d}(G_J-k)$. If Player $J-k+1$ connects to Node $J-k$, all nodes after $J-k+1$ will connect to node $J-k$, because by then $J-k$ will be the only node in $\mathbf{d}(G_t)$.

Player $J-k$ will receive $1 + \delta(J-2) - C$ from playing myopically and $p(J-1) + (1-p)(J-k-1+\delta k) - (J-k-1)C$ from vying. In both cases the first term (of the form ...C) represents the costs of connections. The first terms describe the expected benefit from connections.

Player $J-k$ will then choose to vie if

$$p(1-\delta)(J-2) + (1-p)(J-k)(1-\delta) > (J-k)C$$

If this relation is not satisfied when $k = J-3$, then no nodes will vie, and the resulting network will be a star on Node 2. We call the first node that satisfies the condition for vying optimality v_1. Call the set of all optimal vying nodes, $\mathbf{v}$.

Player $v_j - k$: Consider now a Player $v_j - k$ who moves k moves before the next vying node, v_j. For now, we assume that player $v_j + 1$ will not also be vying for dominance. We will revisit this assumption later in the proof.

This time we will write the choice in terms of the gains and costs vying relative to myopic play to reduce extraneous terms. The extra myopic cost (net gains from the immediate connections) is: $\gamma_{v_j-k} = (v_j - k - 2)(C - 1 + \delta)$

The relative future benefit of vying from future direct connections will be $\xi_{v_j-k} = p(k-1)(1-\delta) + (1-p)\xi_{v_j}$.

The term $p(k-1)(1-\delta)$ comes from the fact that with probability p, Player $v_j - k$ will receive a connection from the next myopic player and become the only dominant node until the next node vies. This next term reflects the fact that, conditional on becoming the only dominant node, Player $v_j - k$ will be the second oldest dominant node after Player v_j vies for dominance. Then with probability $1-p$, Player $v_j - k$ will receive a direct connection from Player $v_j - k+1$ and become the only dominant node after period $v_j - k + 1$. The same happens to Player v_j with probability p and provides all his expected benefit from vying. That means that Player $v_j - k$ gets an expected benefit from vying due to connections from after v_j which is equal to $p\frac{1-p}{p}\xi_{v_j} = (1-p)\xi_{v_j}$.

defined recursively, since after Player v_j vies, there is a $1-p$ chance that

Player $v_j - k$ will vie if the gains are greater than the costs. As the time until the next platform increases and ξ_{v_j-k} increases, and as we move backwards towards the first move, γ_i decreases. This means that, given a fixed next vying player, Player v_j, we can find the previous vying player by going backwards until we find a node such that $\gamma_i < \xi_i$.

Define k_j as the lowest value of k such that $\gamma_{v_j-k_j} < \xi_{v_j-k_j}$, and then say $v_j - k_j = v_{j+1}$. See Figure 5 for a visualization of this relationship based on the $p = 1$ case.

Now we ignore the integer constraints on node indices and say that a node will choose to vie when the costs equal the benefits. In other words, we assume v_j satisfies$\gamma_{v_j} = \xi_{v_j} \forall j$. We can see the kind of errors that are introduced by this approximation by comparing the dotted line and the dashed line in Figure 5. The dotted line shows where the relative costs and benefits are equal, while the dashed line shows gives the actual index of Player v_{j+1}. The difference between the dashed line and the dotted line will always be less than one. We discuss the issue of approximation in more detail in the next section.

Recall that

$$\xi_{v_{j+1}} = p(k_j - 1)(1 - \delta) + (1 - p)\xi_{v_j}$$

and note that

$$\gamma_{v_{j+1}} = \gamma_{v_j} - k_j(C - 1 + \delta)$$

Since node v_{j+1} will have k_j fewer nodes to connect to when vying. Ignoring integer constraints, we can substitute in $\xi_{v_{j+1}} = \gamma_{v_{j+1}}$and $\xi_{v_j} = \gamma_{v_j}$ to get

$$\xi_{v_{j+1}} = \xi_{v_j} - k_j(C - 1 + \delta)$$

Which we can plug in for $\xi_{v_{j+1}}$to get

$$\xi_{v_j} - k_j(C - 1 + \delta) = p(k_j - 1)(1 - \delta) + (1 - p)\xi_{v_j}$$

Which simplifies to

$$\xi_{v_j} = k_j(\tfrac{C}{p} - (\tfrac{1}{p} - 1)(1 - \delta)) - (1 - \delta)$$

Which also means

$$\xi_{v_{j+1}} = k_{j+1}(\tfrac{C}{p} - (\tfrac{1}{p} - 1)(1 - \delta)) - (1 - \delta)$$

If we plug these both back in, we get

$$k_{j+1}(\tfrac{C}{p} - (\tfrac{1}{p} - 1)(1 - \delta)) - (1 - \delta) = p(k_j - 1)(1 - \delta) + (1 - p)k_j(\tfrac{C}{p} - (\tfrac{1}{p} - 1)(1 - \delta)) - (1 - p)(1 - \delta)$$

Which simplifies to

$$k_{j+1} = k_j \tfrac{C(1-p)-(1-2p)(1-\delta)}{C-(1-p)(1-\delta)}$$

Define

$$\lambda = \tfrac{C(1-p)-(1-2p)(1-\delta)}{C-(1-p)(1-\delta)}$$

So we have

$$k_{j+1} = \lambda k_j$$

Note that $\lambda < 1$ as long as $C > 1 - \delta$. Therefore

$$k_j = \lambda^{j-1} k_1$$

Demonstrating an exponential shrinking of the time between vying nodes as we go back in time (hence exponential growth as one moves forward in time).

We can finish the solution by finding and plugging in k_1, which is defined by

$$pk_1(1 - \delta) = (J - k_1 - 1)(C - 1 + \delta)$$

or

$$k_1 = \tfrac{J-1}{p}\left(1 - \tfrac{1-\delta}{C}\right)$$

Which gives our solution

$$k_j = \lambda^{j-1} \tfrac{J-1}{p}\left(1 - \tfrac{1-\delta}{C}\right)$$

Figure 6 shows graphically how this geometric relationship arises when $p = 1$.

Note that the logic we apply here works as long as no two players vie in a row. If two players vie in a row, that immediately means earlier vying nodes cannot receive any further connections from myopic players. In other words it can continue until a value $k_j \leq 1$. Once that occurs, two players will vie in a row, and we will essentially start the process over, since the double vie behaves the same as the end of the game in terms of determining the future benefits of vying for dominance.

Thus if we continue the backwards induction reasoning we will still see an exponential shrinking of the time between vying nodes, but the the exponential shrinking will "reset" occasionally, so it will not be consistent. However, we can guarantee consistent exponential expansion as long as we are only considering the part of the process that takes place after the final time two nodes vie in a row. We can find the last time to nodes vie in a row by looking at $\tilde{j}$ which we define as the smallest value of j such that

$$\lambda^{j-1}\frac{J-1}{p}\left(1-\frac{1-\delta}{C}\right) \leq 1$$

To figure out which node will be the first to of the pair of nodes that vie in a row we simply have to sum the k_js up through $k_{\tilde{j}}$. Thus we define

$$\bar{j} = \max\left(J - \sum_{j=1}^{\tilde{j}} \lambda^{j-1}\frac{J-1}{p}\left(1-\frac{1-\delta}{C}\right), 0\right)$$

We need to include the max here, since it is possible that no two players will ever vie in a row. Thus, after $\bar{j}$, the time between vying players will increase exponentially over time. $\square$

Two questions naturally arise regarding the quality of the approximation. First, how high is $\bar{j}$? Second, how good is the approximation?

Part 3: Next we consider how large the errors introduced by our decision to ignore integer constraints might be. Define the true vying players $\bar{v}_j$ and the approximate v_j derived by ignoring integer node indexing constraints. We also define $\bar{k}_j$ by $\bar{v}_j = \bar{v}_{j-1} - \bar{k}_j$ and true vying benefit $\bar{\xi}_{v_{j+1}} = p(\bar{k}_j - 1)(1-\delta) + (1-p)\bar{\xi}_{v_j}$. Begin by noting that $\bar{v}_1$ will be less than 1 away from the approximate v_1, because $v_1 = \frac{J-1}{p}\left(1-\frac{1-\delta}{C}\right)$, and $\bar{v}_1 = truncate\left(\frac{J-1}{p}\left(1-\frac{1-\delta}{C}\right)\right)$ (see Figure 5 for a visualization of the relationship between the true value and the approximation).

In other words$|\bar{v}_1 - v_1| < 1$. In addition, note that $v_1 \geq \bar{v}$ which means that $\bar{\xi}_{v_1} - \xi_{v_1} \in [0, p(1-\delta))$. Now take as given a true value $\bar{v}_j$ with the approximation v_j and assume $v_j \geq \bar{v}_j$.

We know

$$(v_{j-1} - 1)(C - 1 + \delta) - k_j(C - 1 + \delta) = p(k_j - 1)(1-\delta) + (1-p)\xi_{v_{j-1}}$$

We can derive the analogous relationship for $\bar{k}_j$

$$\bar{k}_j = roundup\left(\frac{(\bar{v}_{j-1}-1)(C-1+\delta)+p(1-\delta)-(1-p)\bar{\xi}_{v_{j-1}}}{C-(1-p)(1-\delta)}\right)$$

Combine these to get

$$\bar{k}_j - k_j < \frac{(\bar{v}_{j-1}-v_{j-1})(C-1+\delta)+(1-p)(\bar{\xi}_{v_{j-1}}-\xi_{v_{j-1}})}{C-(1-p)(1-\delta)} + 1$$

Where the +1 at the end comes from the rounding.

We also have the following for true values

$$\bar{v}_j = \bar{v}_{j-1} - \bar{k}_j$$

$$\bar{\xi}_{v_j} = p(\bar{k}_j - 1)(1-\delta) + (1-p)\bar{\xi}_{v_{j-1}}$$

As well as the same relationship for approximated values

$$v_j = v_{j-1} - k_j$$

$$\xi_{v_j} = p(k_j - 1)(1-\delta) + (1-p)\xi_{v_{j-1}}$$

Which can be combined

$$\bar{v}_j - v_j = (\bar{v}_{j-1} - v_{j-1}) - (\bar{k}_j - \bar{k}_j)$$

$$\bar{\xi}_{v_j} - \xi_{v_j} = p(1-\delta)(\bar{k}_j - k_j) + (1-p)(\bar{\xi}_{v_{j-1}} - \xi_{v_{j-1}})$$

Which can be combined with the worst case relation

$$\bar{k}_j - k_j = (\bar{v}_{j-1} - v_{j-1})(C - 1 + \delta) + (1-p)(\bar{\xi}_{v_{j-1}} - \xi_{v_{j-1}}) + 1$$

To create a system of difference equations

$$\bar{v}_j - v_j = (\bar{v}_{j-1} - v_{j-1}) - \frac{(\bar{v}_{j-1} - v_{j-1})(C-1+\delta) + (1-p)(\bar{\xi}_{v_{j-1}} - \xi_{v_{j-1}})}{C-(1-p)(1-\delta)} - 1$$

or

$$\bar{v}_j - v_j = (\bar{v}_{j-1} - v_{j-1})\left(\frac{p(1-\delta)}{C-(1-p)(1-\delta)}\right) - \frac{(1-p)}{C-(1-p)(1-\delta)}(\bar{\xi}_{v_{j-1}} - \xi_{v_{j-1}}) - 1$$

and

$$\bar{\xi}_{v_j} - \xi_{v_j} = p(1-\delta)\left(\frac{(\bar{v}_{j-1} - v_{j-1})(C-1+\delta) + (1-p)(\bar{\xi}_{v_{j-1}} - \xi_{v_{j-1}})}{C-(1-p)(1-\delta)} + 1\right) + (1-p)(\bar{\xi}_{v_{j-1}} - \xi_{v_{j-1}})$$

or

$$\bar{\xi}_{v_j} - \xi_{v_j} = p(1-\delta)\left(\frac{(C-1+\delta)}{C-(1-p)(1-\delta)}\right)(\bar{v}_{j-1} - v_{j-1}) + (1-p)\left(\frac{C-(1-p)(1-\delta)+1}{C-(1-p)(1-\delta)}\right)(\bar{\xi}_{v_{j-1}} - \xi_{v_{j-1}}) + p(1-\delta)$$

Which combine to give

$$\begin{bmatrix} \bar{\xi}_{v_j} - \xi_{v_j} \\ \bar{v}_j - v_j \end{bmatrix} = \begin{bmatrix} (1-p)\frac{C-(1-p)(1-\delta)+1}{C-(1-p)(1-\delta)} & p(1-\delta)\left(\frac{(C-1+\delta)}{C-(1-p)(1-\delta)}\right) \\ \frac{(1-p)}{C-(1-p)(1-\delta)} & \frac{p(1-\delta)}{C-(1-p)(1-\delta)} \end{bmatrix} \begin{bmatrix} \bar{\xi}_{v_{j-1}} - \xi_{v_{j-1}} \\ \bar{v}_{j-1} - v_{j-1} \end{bmatrix} + \begin{bmatrix} p(1-\delta) \\ -1 \end{bmatrix}$$

This system can be solved using standard methods for matrix difference equations, but the answer is not particularly parsimonious. It is likely more illustrative to simply bound the errors based on the largest eigenvalue of the transition matrix. We know that $|\bar{v}_j - v_j| < |x_j|$ where x_j is defined by $x_1 = -1$ and $x_{j+1} = \lambda_{max} x_j - 1$.

By the Perron Frobenious Theorem, the largest eigenvalue of a matrix is bounded by the largest row sum of that matrix. Therefore

$$\gamma_{max} \leq \max\left(\frac{C+\delta-C\delta p-\delta^2 p-p^2+\delta p^2}{C-(1-p)(1-\delta)}, \frac{1-p\delta}{C-(1-p)(1-\delta)}\right)$$

If $\gamma_{max} < 1$ then $|\bar{v}_j - v_j|$ must converge to some steady state value of $\frac{1}{1-\gamma_{max}}$. If $\gamma_{max} = 1$ then it is possible that the degree of error will not converge. In that case $|\bar{v}_j - v_j|$ will still be bounded above by j. Finally if $\gamma_{max} > 1$, it is possible that the approximation error will explode at a faster than linear rate. In all cases, the approximation is generally more accurate near the end of the game. It should be noted that when p is high, the approximation error is generally well behaved, because when p goes to one, the difference relation governing the evolution of $\bar{v}_j - v_j$ becomes.

$$\bar{v}_j - v_j = \frac{(1-\delta)}{C}(\bar{v}_{j-1} - v_{j-1}) - 1$$

and in all non-trivial cases $\frac{(1-\delta)}{C} < 1$. In addition, when p is high, it becomes less common for two players in a row to vie for dominance. In fact it is impossible for two players to vie in a row if

$p = 1$. This means we only need to consider the rounding error in this case. In this case the solution simplifies to.

$$k_j \approx \left(\tfrac{1-\delta}{C}\right)^{j-1} (J-1)\left(1 - \tfrac{1-\delta}{C}\right)$$

and the difference between the approximation and true value for k_j will always be less than j.